Lecture 27 (6/5/2013)

Theorem 1: The eigen values of a matrix $A_{n\times n}$ are the roots of the characteristic poly. of $A_{n\times n}$

Proof: Let λ be an eigen value of $A_{n\times n}$. Then
$AX = \lambda X \Rightarrow \lambda X - AX = 0 \Rightarrow (\lambda I_n - A_{n\times n})X = 0$
This system has a nontrivial solution iff $|\lambda I_n - A| = 0$

So $f(\lambda) = |\lambda I_n - A| = 0 \Rightarrow \lambda$ is a solution of the characteristic poly $f(\lambda) = \lambda^n - a_1\lambda^{n-1} + \cdots + a_n$
Conversely suppose that λ is a root of the characteristic poly of A. Then $f(\lambda) = 0 = |\lambda I_n - A|$

$|\lambda I_n - A| = 0 \Rightarrow (\lambda I_n - A)X = 0$ has the trivial solution $\Rightarrow \lambda I_n X - AX = 0$.
$\lambda X - AX = 0 \Rightarrow \lambda X = AX$ or
$AX = \lambda X \Rightarrow \lambda$ is an eigen value of $A_{n\times n}$.

Def 2 $A_{n\times n}$ is similar to $B_{n\times n}$ if there is $P_{n\times n}$ nonsingular matrix s.t. $B_{n\times n} = P^{-1}_{n\times n} A P_{n\times n}$.
Similar is an eq. relation!
1st note that $A_{n\times n} = I_n^{-1} A_{n\times n} I_n$ so $A_{n\times n}$ is similar to itself.
Suppose that $A_{n\times n}$ is similar to $B_{n\times n} \rightarrow B_{n\times n} = P^{-1}_{n\times n} A_{n\times n} P_{n\times n}$

$\Rightarrow P_{n\times n} B_{n\times n} = A_{n\times n} P_{n\times n} \Rightarrow A_{n\times n} = P_{n\times n} B_{n\times n} P^{-1}_{n\times n}$

$\Rightarrow A_{n\times n} = (P^{-1}_{n\times n})^{-1} B_{n\times n} P^{-1}_{n\times n}$. So $B_{n\times n}$ is similar to $A_{n\times n}$.
So similar is a symm. relation

Now suppose that $A_{n\times n}$ is similar to $B_{n\times n}$ and

$B_{n\times n}$ is similar to $C_{n\times n}$, then $A_{n\times n}$ is similar to $C_{n\times n}$. Now $A_{n\times n}$ is similar to $B_{n\times n} \Rightarrow B = P^{-1}_{n\times n} A_{n\times n} P_{n\times n}$ and $B_{n\times n}$ is similar to $C_{n\times n}$ then $C_{n\times n} = Q^{-1}_{n\times n} B_{n\times n} Q_{n\times n}$ where $P_{n\times n}, Q_{n\times n}$ are nonsingular matrices. Then

$$C_{n\times n} = Q^{-1}_{n\times n} B_{n\times n} Q_{n\times n} = Q^{-1}_{n\times n} P^{-1}_{n\times n} A_{n\times n} P_{n\times n} Q_{n\times n}$$

$$= (PQ)^{-1}_{n\times n} A_{n\times n} (PQ)_{n\times n}$$ as P, Q are nonsingular

then PQ is nonsingular. So $A_{n\times n}$ is similar to $C_{n\times n}$ i.e. similar is a transitive relation $\Rightarrow$ similar is an equivalence relation.

<u>Def 3</u> $A_{n\times n}$ is a diagonalizable if $A_{n\times n}$ is similar to a diagonal matrix $D_{n\times n}$ i.e. there is a nonsingular matrix $P_{n\times n}$ s.t. $P^{-1}_{n\times n} A_{n\times n} P_{n\times n} = D_{n\times n}$ a diagonal matrix

<u>Theorem 4</u>: $A_{n\times n}$ is a diagonalizable matrix iff it has n lin. indep eigen vectors corresp. to the n eigen values of $A_{n\times n}$

<u>Theorem 5</u>: A matrix $A_{n\times n}$ is diagonalizable if all the roots of its characteristic poly are real and distinct

note that if all the roots of the charact. poly are real but not distinct then the matrix may or may not diagonalizable.

But as we said if $f(\lambda) = (\lambda-\lambda_1)^{k_1} \dots (\lambda-\lambda_r)^{k_r}$ is the charac. poly and for each λ_i, λ_i has k_i lin. indep eigen vectors $\forall i = 1, 2, \dots, r$ (where $k_1 + k_2 + \dots + k_r = n$) then $A_{n\times n}$ is a diagonalizable matrix.

Def 6 A nonsingular matrix $A_{n\times n}$ is said to be an orthogonal matrix if $A^{-1}_{n\times n} = A^T_{n\times n}$

so $A^T A = A A^T = I_n$

Recall that if $A_{n\times n}$ is diagonalizable then there is $P_{n\times n}$ a nonsingular matrix s.t.

$P^{-1}_{n\times n} A_{n\times n} P_{n\times n} = D_{n\times n}$ where here

$$P_{n\times n} = \begin{bmatrix} X_1 & X_2 & \dots & X_n \end{bmatrix} \text{ and } D_{n\times n} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & & \\ 0 & \dots & & \lambda_n \end{bmatrix}$$

where $\{X_1, X_2, \dots, X_n\}$ is a lin indep set of eigen vectors corresp to the eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$ of $A_{n\times n}$.

Note that if $A_{n\times n}$ is an orthogonal matrix then

$A^T A_{n\times n} = A_{n\times n} A^T = I_n \Rightarrow |A_{n\times n} A^T| = |A_{n\times n}| |A^T_{n\times n}| = 1$

$|A_{n\times n}| |A^T| = 1 \Rightarrow |A_{n\times n}|^2 = 1 \Rightarrow |A_{n\times n}| = \pm 1$

$(|A^T| = |A|)$

Theorem 7: Let $A_{n\times n}$ be an $n\times n$ matrix then $A_{n\times n}$

is orthogonal iff the columns of A form an orthonormal basis of $\mathbb{R}^n$.

Theorem 8: Let $A_{n\times n}$ be a symm. matrix then the roots of $A_{n\times n}$ are real nos.

Corollary 9: If $A_{n\times n}$ is a symm matrix s.t. all the roots are distinct roots of the charact. poly then $A_{n\times n}$ is a diagonalizable.

Theorem 10: If $A_{n\times n}$ is a symm matrix then the eigen vectors that belongs to distinct eigen values of $A_{n\times n}$ are orthogonal.

Ex1: Show that the eigen values of A and A^T are exactly the same.

Proof: $|\lambda I_n - A^T| = |(\lambda I_n - A)^T| = |\lambda I_n - A| = f(\lambda)$, so the eigen values of A and A^T are exactly the same

Ex2 Show that if $A_{n\times n}$ is similar to $B_{n\times n}$ then $A_{n\times n}$ and $B_{n\times n}$ have exactly the same eigen values

$A_{n\times n}$ is similar to $B_{n\times n}$ so $B_{n\times n} = P^{-1}AP$ for some nonsingular matrix $P_{n\times n}$.

$|\lambda I_n - B| = |\lambda I_n - P^{-1}AP| = |P^{-1}\lambda I_n P - P^{-1}AP|$

$|P^{-1}(\lambda I_n P - AP)| = |P^{-1}|\,|\lambda I_n P - AP|$

$= |P^{-1}|\,|(\lambda I_n - A)P| = |P^{-1}|\,|\lambda I_n - A|\,|P| = |\lambda I_n - A| = f(\lambda)$

Ex 3 Suppose that $A_{n\times n}$ is an upper triangular matrix then the eigen values of $A_{n\times n}$ are the elements on the main diagonal of $A_{n\times n}$

Proof: $A_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ 0 & \cdots & & a_{nn} \end{bmatrix}$

$f(\lambda) = |\lambda I_n - A|_{n\times n} = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ 0 & \lambda - a_{22} & -a_{12} \cdots & -a_{2n} \\ \vdots & & & \\ 0 & 0 & 0 \cdots & \lambda - a_{nn} \end{vmatrix}$

$= (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn})$

$f(\lambda) = 0 = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn})$

$\Rightarrow \lambda = a_{11}, a_{22}, \cdots, a_{nn}$

Remark $A_{n\times n}$ is nilpotent iff $A^m_{n\times n} = 0$ for some +ve integer m

Ex 4 Let $A_{n\times n}$ be a nilpotent matrix, show that the only eigen value of $A_{n\times n}$ is $\lambda = 0$.

Let λ be an eigen value of A so. Suppose that $A^m = 0$

$m \in \mathbb{Z}^+ \Rightarrow A^m_{n\times n} X = \lambda^m X \Rightarrow 0X = 0 = \lambda^m X, \; X \neq 0$

(X is an eigen vector corresp to λ so $X \neq 0$)

$\Rightarrow \lambda^m = 0 \Rightarrow \lambda = 0$.

Ex 5 Is $|A_{n\times n}| = \lambda_1 \lambda_2 \cdots \lambda_n$ (λ_i's are the roots of charact. Poly.) Then 0 is an eigen value of A iff $A_{n\times n}$ is a singular matrix

Suppose that 0 is an eigen value of $A_{n\times n}$ so $\lambda_i = 0$ for some i

$\Rightarrow \lambda_1 \lambda_2 \cdots \lambda_i \cdots \lambda_n = 0 \Rightarrow 0$ is a factor of the product $\lambda_1 \lambda_2 \cdots \lambda_n = |A| \Rightarrow |A_{n\times n}| = 0 \Rightarrow A_{n\times n}$ is a nonsingular matrix

Conv suppose that $A_{n\times n}$ is a singular matrix

$\Rightarrow |A_{n\times n}| = 0 \Rightarrow 0 = \lambda_1 \lambda_2 \cdots \lambda_n \Rightarrow \lambda_j = 0$ for some j

<u>Ex5</u>: Let $P^{-1}_{n\times n} A_{n\times n} P_{n\times n} = B_{n\times n}$ for some nonsingular matrix $P_{n\times n}$. Let $0 \neq X$ be an eigen vector of A corresp to the eigen value λ of A. Show that $P^{-1}X$ is an eigen vector of B

Now $AX = \lambda X$. Now $P^{-1}_{n\times n} A_{n\times n} P_{n\times n} = B_{n\times n}$

$\Rightarrow A_{n\times n} = P B_{n\times n} P^{-1}_{n\times n}$

$AX = (P B P^{-1}) X = \lambda X$

$B(P^{-1}X) = P^{-1}\lambda X = \lambda (P^{-1}X)$

so $P^{-1}X$ is an eigen vector of B corresp to λ.

<u>Ex6</u> Let $A_{n\times n}, B_{n\times n}$ be two nonsingular matrices

Then AB, and BA are similar $B^{-1}(BA)B = AB$

So AB and BA are similar matrices

Ex7 : $A_{n\times n}$ is a diagonalizable and $A_{n\times n}$ is nonsingular then A^{-1} is diagonalizable

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & \lambda_n \end{bmatrix}$$

$$(P^{-1}AP)^{-1} = D^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{\lambda_n} \end{bmatrix}$$

$$P^{-1}A^{-1}P = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & \frac{1}{\lambda_n} \end{bmatrix} = D^{-1}$$ a diagonal matrix

So A^{-1} is diagonalizable.

Ex8 Show that if $A_{n\times n}$ is diagonalizable then A^T is diagonalizable

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & \lambda_n \end{bmatrix}$$

$$(P^{-1}AP)^T = D^T = D$$

$P^T A^T (P^{-1})^T$

$P^T A^T (P^T)^{-1} = D =$ a diagonal matrix, so

A^T is diagonalizable matrix

Ex9 Suppose that for $0 \neq X$, $A_{n\times n}X = \lambda X$, $B_{n\times n}X = \mu X$ then

$$(A+B)X = AX + BX = \lambda X + \mu X = (\lambda+\mu)X$$

so $\lambda+\mu$ is an eigen value of $A+B$

$(AB)X = A(BX) = A(\mu X) = (A\mu)X$

$= (\mu A)X = \mu(AX) = \mu(\lambda X) = (\mu\lambda)X$

so $\mu\lambda$ is an eigen value of AB

Ex10 Show that if $A_{n\times n}$ is diagonaliza[ble] then $A^k_{n\times n}$ is diagonalizable, $k\in\mathbb{Z}^+$

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & \lambda_n \end{bmatrix}$$

$$(P^{-1}AP)^k = D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & \lambda_n^k \end{bmatrix}$$

$$(P^{-1}AP)^k = \underbrace{P^{-1}AP\,P^{-1}AP \cdots P^{-1}AP}_{k \text{ times}}$$

$= P^{-1}A^2P\,P^{-1}AP \cdots P^{-1}AP$

$= P^{-1}A^kP$

so $(P^{-1}AP)^k = D^k =$ a diagonal matrix

$\Rightarrow (P^{-1}AP)^k$ is a diagonalizable matrix

Ex11 Let $S, T : V \longrightarrow W$ be lin. trans. Then $S+T$ is a lin. trans. where $(S+T)(X) = S(X) + T(X)$

$$(S+T)(X+Y) = S(X+Y) + T(X+Y)$$
$$= S(X) + S(Y) + T(X) + T(Y)$$
$$= S(X) + T(X) + S(Y) + T(Y)$$
$$= (S+T)(X) + (S+T)(Y)$$
$$(S+T)(cX) = S(cX) + T(cX)$$
$$= c\,S(X) + c\,T(X)$$
$$c((S+T)(X))$$

So $S+T$ is a lin. trans.

Show that

(1) $(0+S) = (S+0) = S$

(2) $S+(-S) = (-S)+S = 0$

(3) $((S+T)+L) = S+(T+L)$

(4) If $A(V,W) = \{S : V \to W$ a lin. trans$\}$

Show that $(A(V,W), +)$ is a comm. gp

If $S : V \longrightarrow W$, $T : W \longrightarrow M$

Show that $T \circ S$ is a lin trans where

$$(T\circ S)(X) = T(S(X))$$
$$(T\circ S)(X+Y) = T(S(X+Y)) = T(S(X)+S(Y))$$
$$T(S(X)) + T(S(Y)) = (T\circ S)(X) + (T\circ S)(Y)$$

$$(T \circ S)(cX) = T(S(cX)) = T(c(S(X)))$$
$$= c\,T(S(X))$$
$$= c\,(T \circ S)(X)$$

So $T \circ S$ is a lin. trans.