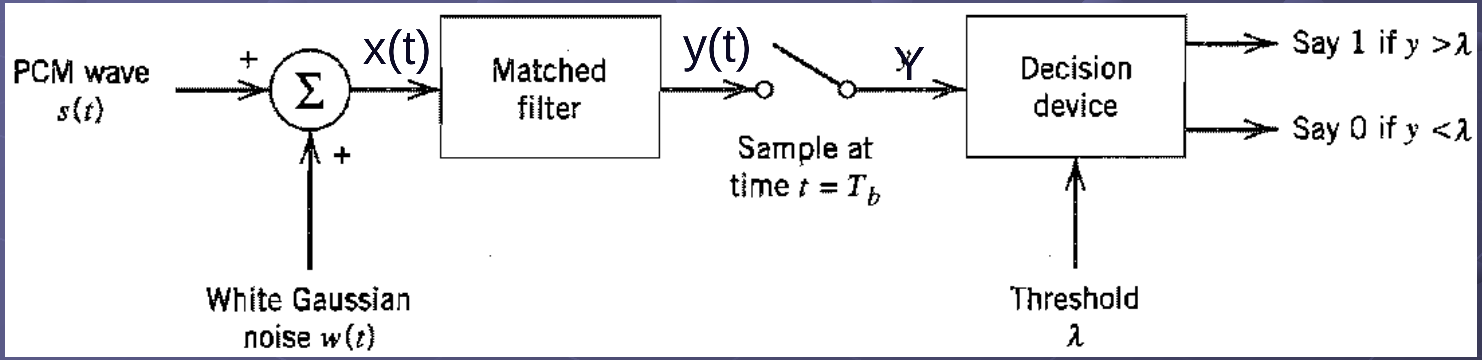
Error rate due to noise

- In this section, an expression for the probability of error will be derived
- The analysis technique, will be demonstrated on a binary PCM based on polar NRZ signaling
- The channel noise is modeled as additive white Gaussian noise $w(t)$ of zero mean and power spectral density $\frac{N_0}{2}$

Bit error rate

- The receiver of the communication system can be modeled as shown below



- The signaling interval will be $0 \le t \le T_b$, where T_b is the bit duration

Bit error rate

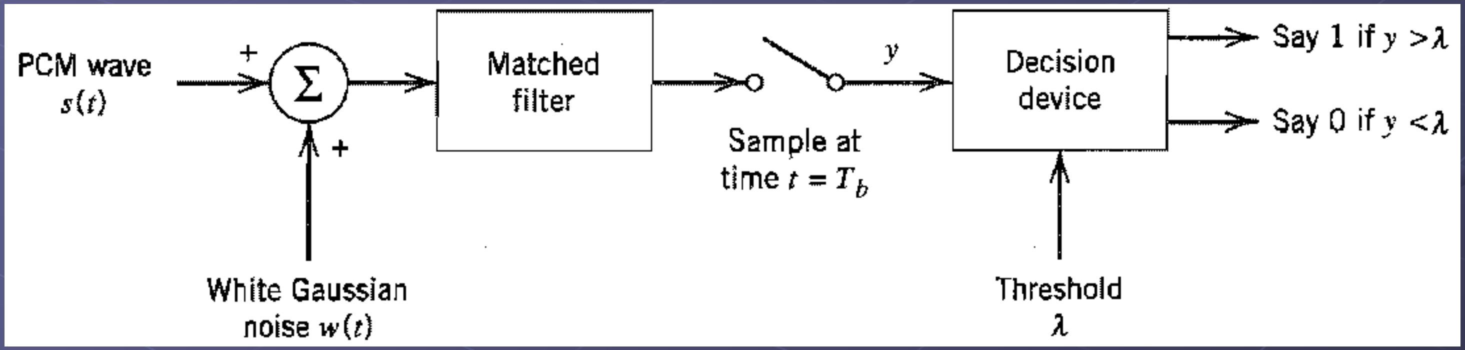
- The received signal can be given by

$$x(t) = \begin{cases} +A + w(t), & symbol\ 1\ was\ sent \\ -A + w(t), & symbol\ 0\ was\ sent \end{cases}$$

- It is assumed that the receiver has acquired knowledge of the starting and ending times of each transmitted pulse
- The receiver has a prior knowledge of the pulse shape, but not the pulse polarity

Receiver model

- The structure of the receiver used to perform this decision making process is shown below



Possible errors

- When the receiver detects a bit there are two possible kinds of error
  1. Symbol 1 is chosen when a 0 was actually transmitted; we refer to this error as an error of the first kind
  2. Symbol 0 is chosen when a 1 was actually transmitted; we refer to this error as an error of the second kind

Average probability of error calculations

- To determine the average probability of error, each of the above two situations will be considered separately
- If symbol 0 was sent then, the received signal is

$$x(t) = -A + w(t), 0 \le t \le T_b$$

Average probability of error calculations

- The matched filter output will be given by

$$y = \frac{1}{T_b}\int_0^{T_b} x(t)dt$$

$$y = -A + \frac{1}{T_b}\int_0^{T_b} w(t)dt = -\mathrm{A}$$

- The matched filter output represents a random variable Y

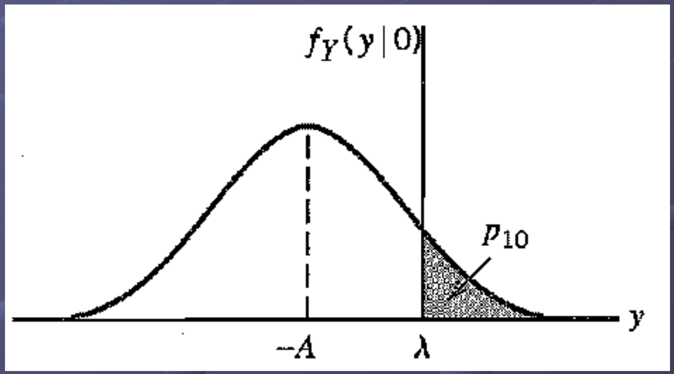
Average probability of error calculations

- This random variable is a Gaussian distributed variable with a mean of $-A$
- The variance of Y is $\sigma^2{}_Y = \frac{N_0}{2T_b}$
- The conditional probability density function the random variable Y, given that symbol 0 was sent, is therefore

$$f_Y(y|0) = \frac{1}{\sqrt{\pi N_0/T_b}} e^{\left(-\frac{(y+A)^2}{N_0/T_b}\right)}$$

Average probability of error calculations

- The probability density function is plotted as shown below



- To compute the conditional probability, p_{10}, of error that symbol 0 was sent, we need to find the area between λ and ∞

- In mathematical notations

$$p_{10} = P(y > \lambda | symbol\ 0\ was\ sent$$

$$p_{10} = \int_{\lambda}^{\infty} f_y(y|0)dy$$

$$p_{10} = \frac{1}{\sqrt{\pi N_0/T_b}} \int_{\lambda}^{\infty} e^{\left(-\frac{(y+A)^2}{N_0/T_b}\right)} dy$$

- The integral

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_{\lambda}^{\infty} e^{(-z^2)} dz$$

- Is known as the complementary error function which can be solved by numerical integration methods

- By letting $z = \frac{y+A}{\sqrt{N_0/T_b}}$, $dy = \sqrt{N_0/T_b}\,dz$, p_{10} became

$$p_{10} = \frac{1}{\sqrt{\pi}} \int_{\left(\frac{A+\lambda}{\sqrt{N_0/T_b}}\right)}^{\infty} e^{-z^2}\,dz$$

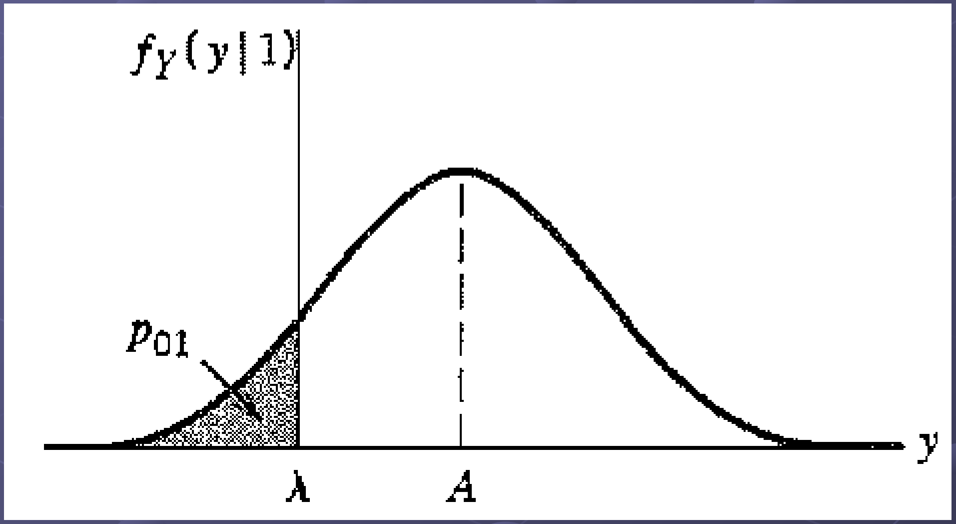
$$p_{10} = \frac{1}{2} erfc\left(\frac{A+\lambda}{\sqrt{N_0/T_b}}\right)$$

- By using the same procedure, if symbol 1 is transmitted, then the conditional probability density function of Y, given that symbol 1 was sent is given

$$f_Y(y|1) = \frac{1}{\sqrt{\pi N_0/T_b}} e^{\left(-\frac{(y-A)^2}{N_0/T_b}\right)}$$

- Which is plotted in the next slide

- The probability density function $f(Y|1)$ is plotted as shown below



- The probability of error that symbol 1 is sent and received bit is mistakenly read as 0 is given by

$$p_{01} = \int_{-\infty}^{\lambda} f_y(y|1)dy$$

$$p_{10} = \frac{1}{\sqrt{\pi N_0/T_b}} \int_{-\infty}^{\lambda} e^{\left(-\frac{(y-A)^2}{N_0/T_b}\right)} dy$$

- By letting $z = \frac{A-y}{\sqrt{N_0/T_b}}$, $dy = -\sqrt{N_0/T_b}\,dz$, p_{10} became

$$p_{10} = -\frac{1}{\sqrt{\pi}} \int_{\infty}^{\frac{A-\lambda}{\sqrt{N_0/T_b}}} e^{-z^2}\,dz = \frac{1}{\sqrt{\pi}} \int_{\frac{A-\lambda}{\sqrt{N_0/T_b}}}^{\infty} e^{-z^2}\,dz$$

$$p_{10} = \frac{1}{2} erfc\left(\frac{A-\lambda}{\sqrt{N_0/T_b}}\right)$$

Expression for the average probability of error

- $$p_{10} = \frac{1}{2} erfc\left(\frac{A-\lambda}{\sqrt{N_0/T_b}}\right)$$

The average probability of error P_e is given by

$$P_e = P_0 P_{10} + P_1 P_{01}$$

$$P_e = \frac{P_0}{2} erfc\left(\frac{A+\lambda}{\sqrt{N_0/T_b}}\right) + \frac{P_1}{2} erfc\left(\frac{A-\lambda}{\sqrt{N_0/T_b}}\right)$$

- In order to find the value of λ which minimizes the average probability of error we need to derive P_e with respect to λ and then equate to zero as

$$\frac{\partial P_e}{\partial \lambda} = 0$$

- From math's point of view (Leibniz's rule)

$$\frac{\partial \operatorname{erfc}(u)}{\partial u} = -\frac{1}{\sqrt{\pi}} e^{-u^2}$$

Optimum value of the threshold λ value

- If we apply the previous rule to P_e, then we may have the following equations

$$\frac{\partial P_e}{\partial \lambda} = -\frac{1}{\sqrt{\pi N_0/T_b}}\frac{p_0}{2}e^{-\frac{(A+\lambda)^2}{N_0/T_b}} + \frac{1}{\sqrt{\pi N_0/T_b}}\frac{p_1}{2}e^{-\frac{(A-\lambda)^2}{N_0/T_b}} = 0$$

- By solving the previous equation we may have

$$\frac{p_0}{p_1} = \frac{e^{-\frac{(A-\lambda)^2}{N_0/T_b}}}{e^{-\frac{(A+\lambda)^2}{N_0/T_b}}} = \frac{f_y(y|1)}{f_y(y|0)}$$

- From the previous equation, we may have the following expression for the optimum threshold point

$$\lambda_{opt} = \frac{N_0}{4AT_b} ln\left(\frac{p_0}{p_1}\right)$$

- If both symbols 0 and 1 occurs equally in the bit stream, then $p_0 = p_1 = \frac{1}{2}$, then

1. $\lambda_{opt} = 0$

- 2. $P_e = \frac{1}{2} erfc\left(\frac{A}{\sqrt{N_0/T_b}}\right)$

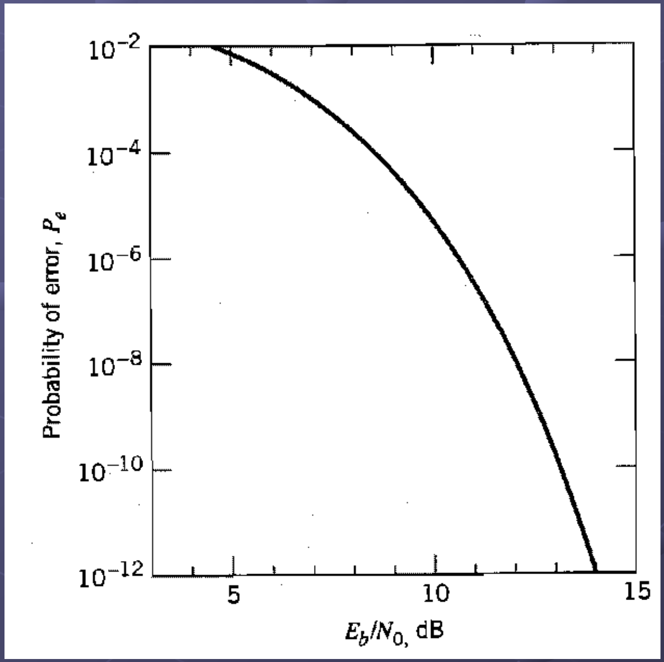
$$= \frac{1}{2} erfc\left(\sqrt{\frac{A^2}{N_0/T_b}}\right)$$

$$= \frac{1}{2} erfc\left(\sqrt{\frac{A^2 T_b}{N_0}}\right)$$

$$= \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Average probability of error vs $\frac{E_b}{N_0}$

- A plot of the average probability of error as a function of the signal to noise energy is shown below



- As it can be seen from the previous figure, the average probability of error decreases exponential as the signal to noise energy is increased
- For large values of the signal to noise energy the complementary error function can be approximated as $$P_e < \frac{\exp(-E_b/N_0)}{2\sqrt{\pi E_b/N_0}}$$

Example1

4.7 A binary PCM system using polar NRZ signaling operates just above the error threshold with an average probability of error equal to 10^{-6}. Suppose that the signaling rate is doubled. Find the new value of the average probability of error. You may use Table A6.6 to evaluate the complementary error function.

Solution

In a binary PCM system, with NRZ signaling, the average probability of error is

$$P_e = \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

The signal energy per bit is

$$E_b = A^2 T_b$$

Example1 solution

where A is the pulse amplitude and T_b is the bit (pulse) duration. If the signaling rate is doubled, the bit duration T_b is reduced by half. Correspondingly, E_b is reduced by half.

Let $u = \sqrt{E_b/N_0}$. We may then set

$$P_e = 10^{-6} = \frac{1}{2}\,\mathrm{erfc}(u)$$

Solving for u, we get

$$u = 3.3$$

When the signaling rate is doubled, the new value of P_e is

$$P_e' = \frac{1}{2}\,\mathrm{erfc}\left(\frac{u}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\,\mathrm{erfc}(2.33)$$

$$= 10^{-3}.$$

Example 2

4.8 A continuous-time signal is sampled and then transmitted as a PCM signal. The random variable at the input of the decision device in the receiver has a variance of 0.01 volts^2.

(a) Assuming the use of polar NRZ signaling, determine the pulse amplitude that must be transmitted for the average error rate not to exceed 1 bit in 10^8 bits.

(b) If the added presence of interference causes the error rate to increase to 1 bit in 10^6 bits, what is the variance of the interference?

Example 2 solution

- The average probability of error is given by $P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$ where $E_b = A^2 T_b$,

We can rewrite $P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{A^2 T_b}{N_0}}\right) = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{A^2}{\frac{N_0}{T_b}}}\right)$

Since $\sigma = \sqrt{\frac{N_0}{2T_b}}$ the above expression for the average probability of error can be expressed as $\mathrm{P_e} = \frac{1}{2}\operatorname{erfc}\left(\frac{A}{\sqrt{2}\sigma}\right)$

Example 2 solution

- Since P_e is very small we can approximate it by $P_e = \frac{1}{2u\sqrt{\pi}} e^{-u^2}$
- For $P_e = 1 \times 10^{-8}$ we have $u = 3.97$
- From the last equation in the previous slide we have $u = \frac{A}{\sqrt{2}\sigma}$
- If $\sigma^2 = 0.01 volt^2 \Rightarrow \sigma = 0.1\, volt$
- $3.97 = \frac{A}{0.01\sqrt{2}} \Rightarrow A = 0.561$

Example 2 solution

- b) Let σ_T denote the combined variance due to noise and inter symbol interference which means $\sigma_T^2 = \sigma_{ISI}^2 + \sigma_{AWGN}^2$

The new value for the average probability of error is

$$P_e = \frac{1}{2}\text{erfc}\left(\frac{A}{\sigma_T\sqrt{2}}\right) = 1 \times 10^{-6}$$

From the erfc table we have $\frac{A}{\sigma_T\sqrt{2}} = 3.36$

Example 2 solution

- $\sigma_T = \frac{A}{u_T\sqrt{2}} = \frac{0.561}{3.36\sqrt{2}} = 0.1181$ volt
- But $\sigma_T^2 = \sigma_{ISI}^2 + \sigma_{AWGN}^2 \Rightarrow \sigma_{ISI}^2 = \sigma_T^2 - \sigma_{AWGN}^2$
- $\sigma_{ISI}^2 = 0.0139 - 0.01 = 0.0039\ volt^2$

Example 3

Consider that NRZ binary pulses are transmitted along a cable that attenuates the signal power by 3 dB (from transmitter to receiver). The pulses are coherently detected at the receiver, and the data rate is 56 kbit/s. Assume Gaussian noise with $N_0 = 10^{-6}$ Watt/Hz. What is the minimum amount of power needed at the transmitter in order to maintain a bit-error probability of $P_B = 10^{-3}$?

Example 3 solution

- Signaling with NRZ pulses represents an example of antipodal signaling, we can use

$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$1 \times 10^{-3} = \frac{1}{2} erfc\left(\sqrt{\frac{A^2 \frac{1}{56000}}{10^{-6}}}\right)$$

Example 3 solution

- Using the $\mathrm{erfc}(x)$ table we find

$$\sqrt{\frac{A^2 \frac{1}{56000}}{10^{-6}}} = 2.18$$

$$A^2 = 0.266$$

- With 3-dB power loss the power at the transmitting end of the cable would be twice the power at the receiving terminal, this means that $P_{tx} = 2P_{rx} = 0.532\ watt$