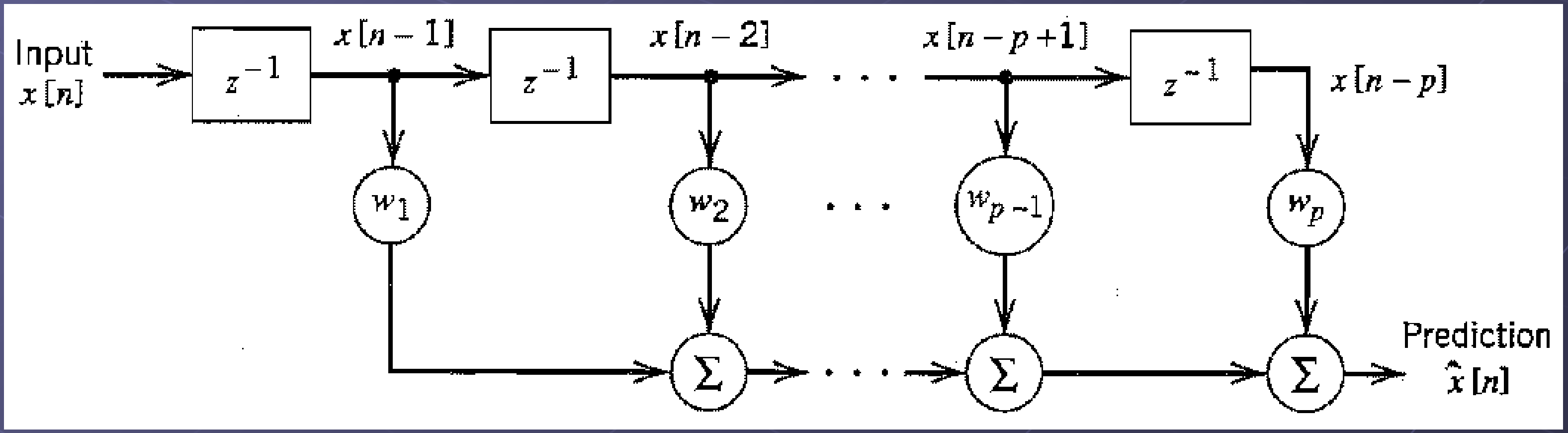
Linear prediction

- Linear prediction is a method used to reduce the bandwidth required to transmit PCM pulses
- It is widely used in speech communications over mobile channels
- In linear prediction the future values of a discrete time signals are estimated as a linear function of previous samples

Linear prediction filter

- The predicted future samples of the discrete signal $\hat{x}[n]$ can be predicted from the present samples of the discrete signals by passing $x[n]$ into the LP filter shown below



Linear prediction filter

- The equation that describes the filter output is given by (convolution)

$$\hat{x}[n] = \sum_{k=1}^{p} w_k x[n-k]$$

- Where *p*, is the number of unit delay elements which is called the prediction order
- w represents the weights of the taps

Linear prediction error

- The prediction error is defined as

$$e[n] = x[n] - \hat{x}[n]$$

- The question that may be asked is how many filter taps are required?
- The design objective is to choose the filter coefficients $w_1 - w_p$ so as to minimize the mean square error

$$J = E\left[e^2[n]\right]$$

Linear prediction error

- If we expand the previous equation for J

$$J = E\left[x^2[n] - 2x[n]\hat{x}[n] - \left(\hat{x}[n]\right)^2\right]$$

$$J = E\left[x^2[n]\right] - 2\sum_{k=1}^{p} w_k E\left[x[n]x[n-k]\right] + \sum_{j=1}^{p}\sum_{k=1}^{p} w_j w_k E\left[x[n-j]x[n-k]\right]$$

- If the input signal $x(t)$ is the sample function of a stationary process $X(t)$ of zero mean, that is, $E\left[x[n]\right] = 0$, then

$$\sigma_x^2 = E\left[x^2[n]\right] - \left(E\left[x[n]\right]\right)^2$$

$$\sigma_x^2 = E\left[x^2[n]\right]$$

Linear prediction error

- The term $E\big[x[n]x[n-k]\big] = R_x[k]$ is the autocorrelation function of $x[n]$
- The mean square error *J* can be rewritten as

$$J = \sigma_x^2 - 2\sum_{k=1}^{p} w_k R_x[k] + \sum_{j=1}^{p}\sum_{k=1}^{p} w_j w_k R_x\big[k-j\big]$$

- In order to find the coefficients which minimizes the error, we differentiate *J* with respect to w_k and equate with zero $\frac{\partial J}{\partial} w_k = 0$

Wiener-Hopf equation

- If we apply the derivative to the mean square error *J* we get the following equation

$$\sum_{j=1}^{p} w_j R_x\left[k-j\right] = R_x[k] = R_x[-k]$$

- The above optimality equation is called the Wiener-Hopf equation for linear prediction
- It is convenient to reformulate the Wiener-Hopf equations in matrix form as follows

Matrix form of Wiener-Hopf equation

$$R_x w_0 = r_x$$

- Where w_0 is p-by-1 optimum coefficient vector $w_0 = \lfloor w_1, w_2, ..., w_p \rfloor$
- r_x is p-by-1 autocorrelation vector

$$r_x = [R_x[1], R_x[2], ..., R_x[p]]$$

Matrix form of Wiener-Hopf equation

- R_x is p-by-p autocorrelation matrix

$$R_x = \begin{bmatrix} R_x[0] & R_x[1] & \cdots & R_x[p-1] \\ R_x[1] & R_x[0] & \cdots & R_x[p-2] \\ \vdots & \vdots & \vdots & \vdots \\ R_x[p-1] & R_x[p-2] & \cdots & R_x[0] \end{bmatrix}$$

Wiener-Hopf equation and the LP filter design

- In order to design the filter we need to compute the tap weights which can be determined from $w_0 = R_x^{-1} r_x$
- The minimum mean-square value of the prediction error is obtained by substituting the values w_o into J which yields the following equation $J \min = \sigma_x^2 - r_x^T R_x^{-1} r_x$

example

3.32 A stationary process $X(t)$ has the following values for its autocorrelation function:

$$R_X(0) = 1$$
$$R_X(1) = 0.8$$
$$R_X(2) = 0.6$$
$$R_X(3) = 0.4$$

(a) Calculate the coefficients of an optimum linear predictor involving the use of three unit-delays.

(b) Calculate the variance of the resulting prediction error.

Solution

Problem 3.32

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0.8 & 0.6 \\ 0.8 & 1 & 0.8 \\ 0.6 & 0.8 & 1 \end{bmatrix}$$

$$\mathbf{r}_x = [0.8,\ 0.6,\ 0.4]^T$$

(a) $\mathbf{w}_0 = \mathbf{R}_x^{-1}\mathbf{r}_x$

$$= \begin{bmatrix} 1 & 0.8 & 0.6 \\ 0.8 & 1 & 0.8 \\ 0.6 & 0.8 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.8 \\ 0.6 \\ 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.875 \\ 0 \\ -0.125 \end{bmatrix}$$

Solution

$$
\begin{aligned}
\text{(b)}\quad J_{\min} &= R_x(0) - \mathbf{r}_x^T \mathbf{R}_x^{-1} \mathbf{r}_x \\
&= R_x(0) - \mathbf{r}_x^T \mathbf{w}_0 \\
&= 1 - \begin{bmatrix} 0.8, & 0.6, & 0.4 \end{bmatrix} \begin{bmatrix} 0.875 \\ 0 \\ -0.125 \end{bmatrix} \\
&= 1 - (0.8 \times 0.875 - \ 0.4 \times \ 0.125) \\
&= 1 - 0.7 + 0.05 \\
&= 0.35
\end{aligned}
$$