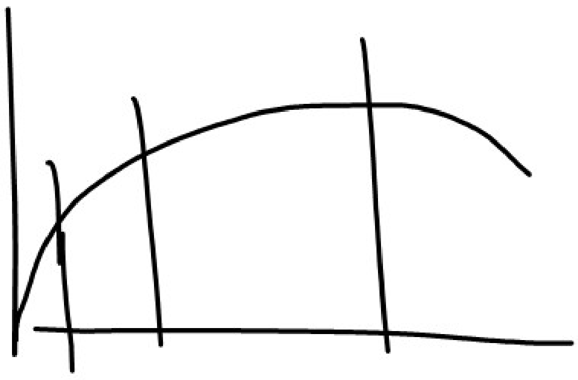
1D $T_n = 2\pi\sqrt{\frac{M}{K}}$

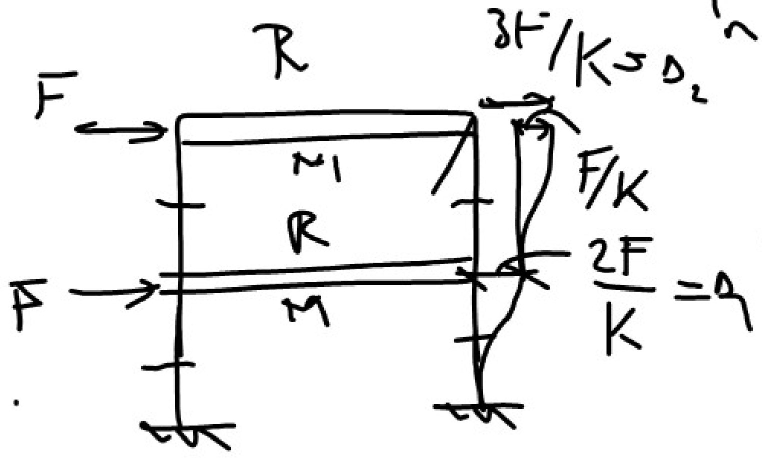
MD Rayleigh $T_n = 2\pi\sqrt{\frac{\sum m_i \Delta_i^2}{\sum F_i \Delta_i}}$

1D $T_n = 2\pi\sqrt{\frac{m_i \Delta_i F_i}{F_i K}}$

1989

$V = ZICKSW$



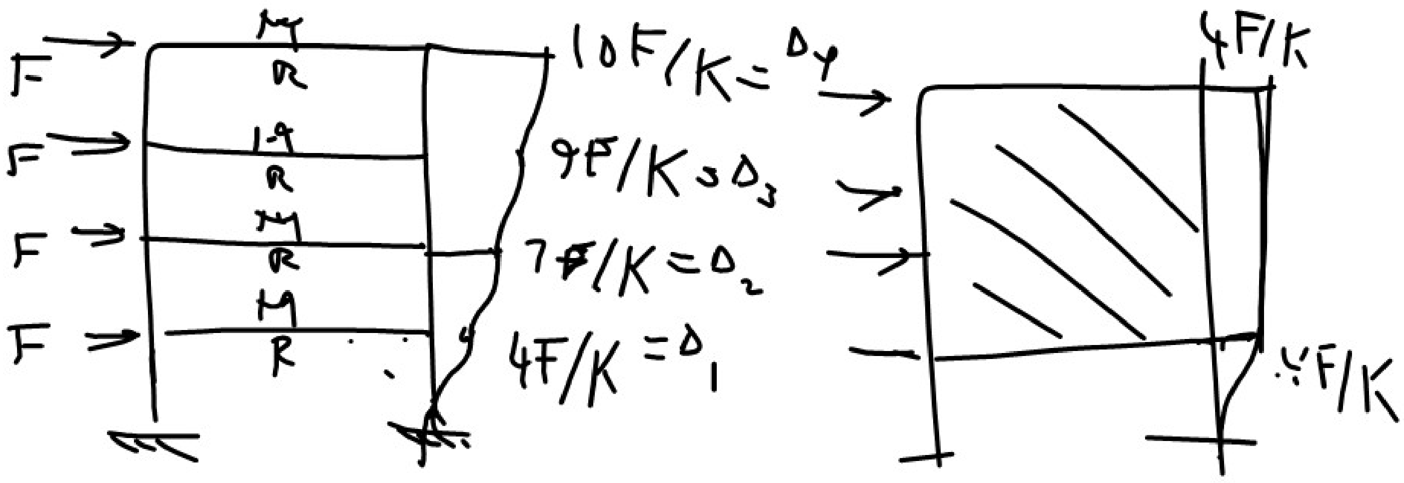


$$T_n = 2\pi\sqrt{\frac{\sum m_i \Delta_i^2}{\sum F_i \Delta_i}}$$

$$= 2\pi\sqrt{\frac{M(2^2+3^2)F^2/K^2}{(2+3)F^2/K}}$$

$$= 2\pi\sqrt{\frac{13M}{5K}}$$ 194 / 8888

$= 1.5$ sec



$$T_n = 2\pi\sqrt{\frac{\sum m_i \Delta_i^2}{\sum F_i \Delta_i}} = 2\pi\sqrt{\frac{M(4^2+7^2+9^2+10^2)F^2/K^2}{(4+7+9+10)F^2/K}}$$

$$= 2\pi\sqrt{\frac{246M}{30K}} = 2.66 \text{ sec}$$

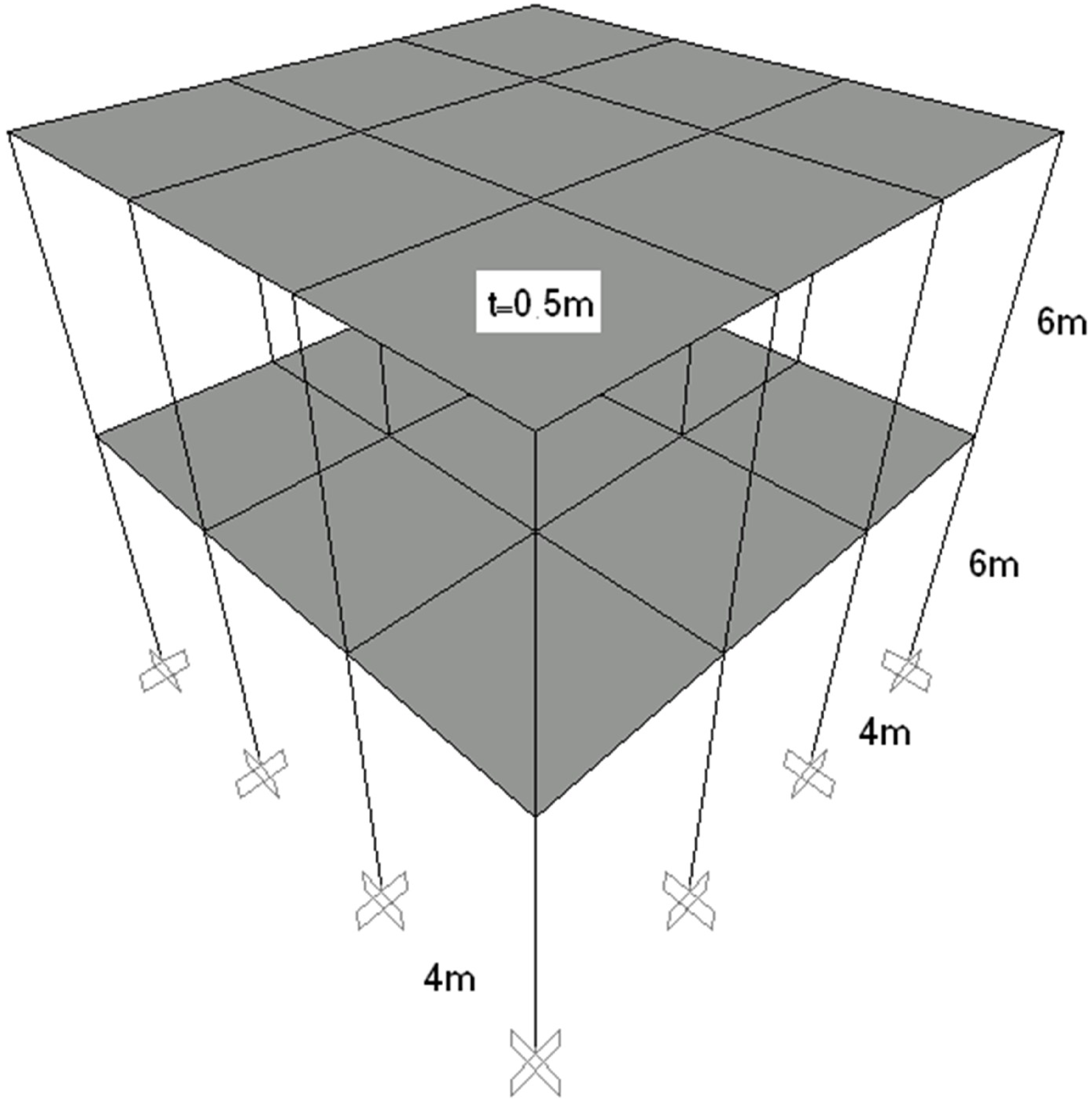
Part 3: Earthquakes/nD

Contents:

- Example Problem: two story structure #1
- Modal Analysis: finite element solution
- Modal Analysis: analogical solution
- Equilibrium differential equations (DE’s)
- Solutions of set of DE’s
  - Homogeneous solution
  - Undamped forced vibrations
  - Damping
- Parameters affecting period calculations

Example Problem:
Two story structure#1

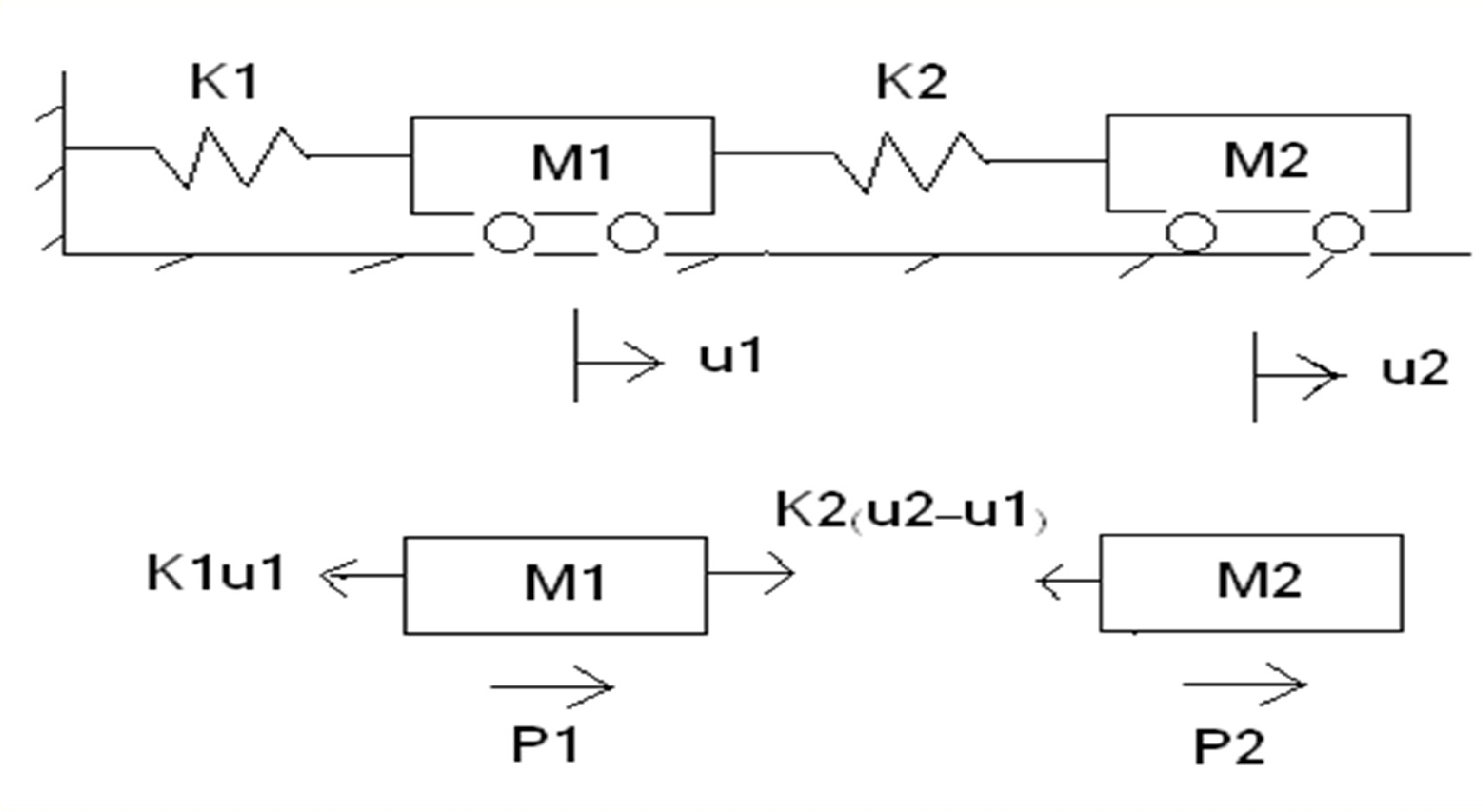
- RC flat plate structure shown next page (same as previous example but two stories).
- Rectangular columns 20cmX60cm
- No superimposed loads
- E=25GPa, μ=0.2, ρ=2.5t/m^3
- Find displacement of first and second floor if structure is subjected to uniform earthquake acceleration 0.3g in y-direction
- Find displacement of first and second floor if structure is subjected to elcentro earthquake in y-direction, then draw response spectrum for acceleration and displacement.


t=0.5m
6m
6m
4m
4m

Modal Analysis: finite element solution

participation direction	Modal mass particip. ratio	period	Mode number
Uy	0.95	1.547	1
Rz	0.93	0.621	2
Uy	0.05	0.594	3
Ux	0.93	0.578	4

Modal Analysis: analogical solution spring mass model



Equilibrium DE's

$$M_1 \ddot{u}_1 = K_2(u_2 - u_1) - K_1 u_1 + P_1$$

$$M_2 \ddot{u}_2 = -K_2(u_2 - u_1) + P_2$$

$$M \ddot{u} + K u = P$$

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, K = \begin{pmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{pmatrix}$$

Solution of set of differential equations

- Solution= Homogeneous + particular
- Homogeneous solution:

$$M\ddot{u} + Ku = 0$$

Homogeneous solution

- Assume

$$u_{1,i} = \varphi_{1,i} \sin(\omega_\iota t + \alpha_i), u_2 = \varphi_{2,i} \sin(\omega_\iota t + \alpha_i)$$

- i=1,2,3…n where n=number of degrees of freedom
- Φ_i=vector of nodal amplitudes (mode shapes) for i-th mode
- ω_i=angular frequency of mode i
- Substitution of above equation into homogeneous differential equations yields

Solution of homogeneous DE's: generalized eigen-problem

$$\left([K] - \omega_i^2 [M]\right)\{\varphi\} = \{0\}$$

Trivial Solution:

$$\left| [K] - \omega_i^2 [M] \right| \neq 0$$

$$\{\Phi\} = 0$$

Nontrivial Solution:

$$\left| [K] - \omega^2 [M] \right| = 0, \{\Phi\} \neq 0$$

$\omega^2{}_i$ ≡ Roots of Characteristic Polynomial (eigenvalues)

$\{\Phi\}_i$ ≡ Associated Eigenvectors

ω_i Natural Frequencies

$\{\Phi\}_i$ Normal Modes

Homogeneous solution

- Assume

$$K_1 = K_2 = K, M_1 = M_2 = M,$$

$$\begin{pmatrix} 2K - \omega_i^2 \mathrm{M} & -K \\ -K & K - \omega_i^2 \mathrm{M} \end{pmatrix} \begin{pmatrix} \varphi_{1,i} \\ \varphi_{2,i} \end{pmatrix} = 0$$

Natural periods

$$\begin{pmatrix} 2K-\omega_\iota^{2}\mathrm{M} & -K \\ -K & K-\omega_\iota^{2}\mathrm{M} \end{pmatrix} = 0$$

$$\mathrm{K}^2 - 3KM\omega_\iota^{2} + M^2(\omega_\iota^{2})^2 = 0$$

$$\omega_1^{2} = .382K/M \rightarrow \omega_1 = 4.18 \rightarrow T_1 = 1.5$$

$$\omega_2^{2} = 2.618K/M \rightarrow \omega_2 = 10.9 \rightarrow T_2 = 0.57$$

Modal Shapes

$$\frac{\varphi_{1,i}}{\varphi_{2,i}} = \frac{K}{2K - \omega_i^2 M}$$

$$\frac{\varphi_{1,1}}{\varphi_{2,1}} = \frac{K}{2K - .382K} = 0.618 = r_1$$

$$\frac{\varphi_{1,2}}{\varphi_{2,2}} = \frac{K}{2K - 2.618K} = -1.618 = r_2$$

$$\varphi = \begin{pmatrix} r_1 & r_2 \\ 1 & 1 \end{pmatrix}$$

Orthogonality Property of modes

- Consider modes i and j

$$K\varphi_i = \omega_i^2 M\varphi_i$$

$$K\varphi_j = \omega_j^2 M\varphi_j$$

- Premultiply first equation by φ_j^T and postmultiply the transpose of second equation by φ_i

$$\varphi_j^T K\varphi_i = \omega_i^2 \varphi_j^T M\varphi_i$$

$$\varphi_j^T K\varphi_i = \omega_j^2 \varphi_j^T M\varphi_i$$

Orthogonality Property of modes

- Subtraction yields

$$0 = (\omega_i^2 - \omega_j^2)\varphi_j^T M \varphi_i$$

- Divide both sides of first equation by ω_i^2 and both sides of second equation by ω_j^2 then do subtraction:

$$\left(\frac{1}{\omega_i^2} - \frac{1}{\omega_j^2}\right)\varphi_j^T K \varphi_i = 0$$

Orthogonality Property of modes

Thus:

$$\{\Phi\}_j^T [M] \{\Phi\}_i = 0$$

$$\{\Phi\}_j^T [K] \{\Phi\}_i = 0$$

$$i \neq j$$

Orthogonality Property of modes

$$\{\Phi\}_i^T [M] \{\Phi\}_i = M_{Pi}$$

$$\{\Phi\}_i^T [K] \{\Phi\}_i = K_{Pi}$$

$$i = j$$

Modal Mass Participation Ratios (MMPR) Derivation

- The total mass of structure may be expressed as

$$M_T = \{1\}^T [M] \{1\}$$

- Where
  - {1) unity column vector
  - [M] mass matrix

MMPR Derivation: Vector Expansion

- Any vector may be expanded in terms of the normal modes, the unity vector may be expanded as:

$$\{1\} = [\Phi]\{\Gamma\}$$

- Where
  - [φ]=the modal matrix, the columns of which are the orthogonal mode shapes
  - {г}=the amplitude of the modes representing the modal participation factors

MMPR Derivation

- Premultiply previous equation by $\{\varphi\}^T$ [M]

$$\{\Phi\}^T[M]\{1\} = \{\Phi\}^T[M][\Phi]\{\Gamma\}$$

- Involving the orthogonality property of the mode shapes:

$$\{\Phi_m\}^T[M]\{\Phi_n\} = 0 \rightarrow m \neq n$$

$$\{\Phi_m\}^T[M]\{\Phi_n\} = M_n \rightarrow m = n$$

- Where M_n is the generalized mass for the n-th mode; thus

$$\{\Phi_n\}^T[M]\{1\} = M_n\Gamma_n = \Psi_n \rightarrow \Gamma_n = \Psi_n / M_n$$

MMPR Derivation

- Thus the starting equation can be written as

$$\{1\} = [\Phi][\Psi_n / M_n]$$

- The expression for the total mass can be written as:
-

$$M_T = \{1\}^T [M]\{1\}$$

$$M_T = \sum_{n=1}^{n} [\Psi_n / M_n]^T [\Phi]^T [M][\Phi][\Psi_n / M_n]$$

$$M_T = \sum_{n=1}^{n} [\Psi_n / M_n]^T M_n [\Psi_n / M_n] = \sum_{n=1}^{n} \Gamma_n^{\,2} M_n$$

Definition of MMPR

- Thus MMPR is defined as

$$MMPR_i = \Gamma_n^{\,2} M_n / M_T$$

Finding MMPR for previous example

- Finding

$$\{\Phi\}^T[M][\Phi]$$

$$= \begin{bmatrix} 0.618 & 1 \\ -1.618 & 1 \end{bmatrix} \begin{bmatrix} 194.4 & 0 \\ 0 & 194.4 \end{bmatrix} \begin{bmatrix} 0.618 & -1.618 \\ 1 & 1 \end{bmatrix}$$

$$\{\Phi\}^T[M][\Phi] = \begin{bmatrix} 268 & 0 \\ 0 & 703 \end{bmatrix}$$

- Then

Finding MMPR for previous example

$$\Gamma_n = \{\Phi_n\}^T [M]\{1\} / M_n$$

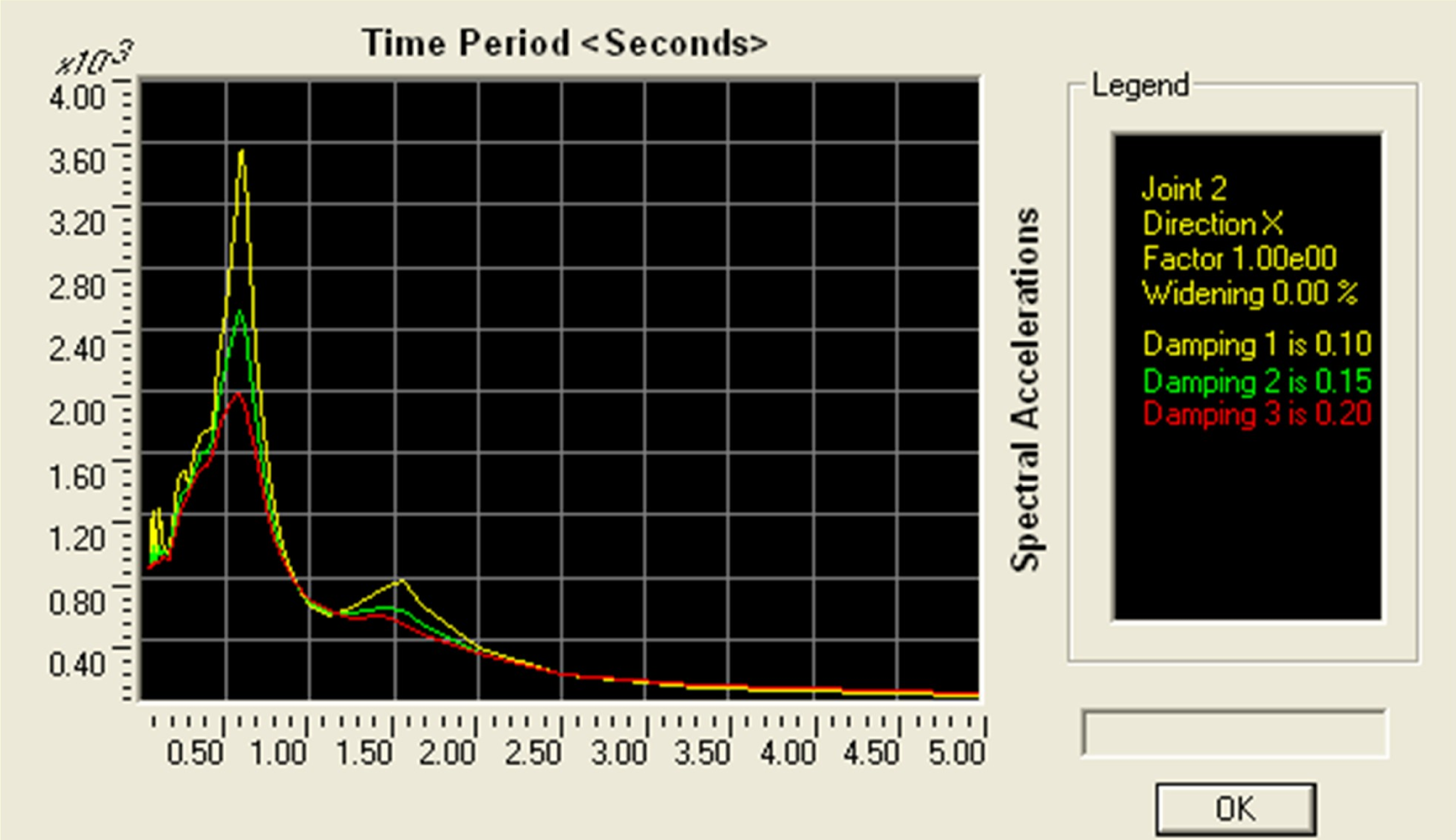
$$\Gamma = \begin{bmatrix} 0.618 & 1 \\ -1.618 & 1 \end{bmatrix} \begin{bmatrix} 194.4 & 0 \\ 0 & 194.4 \end{bmatrix} \begin{bmatrix} \dfrac{1}{268} \\ \dfrac{1}{703} \end{bmatrix} = \begin{bmatrix} 1.17 \\ -0.17 \end{bmatrix}$$

- Then

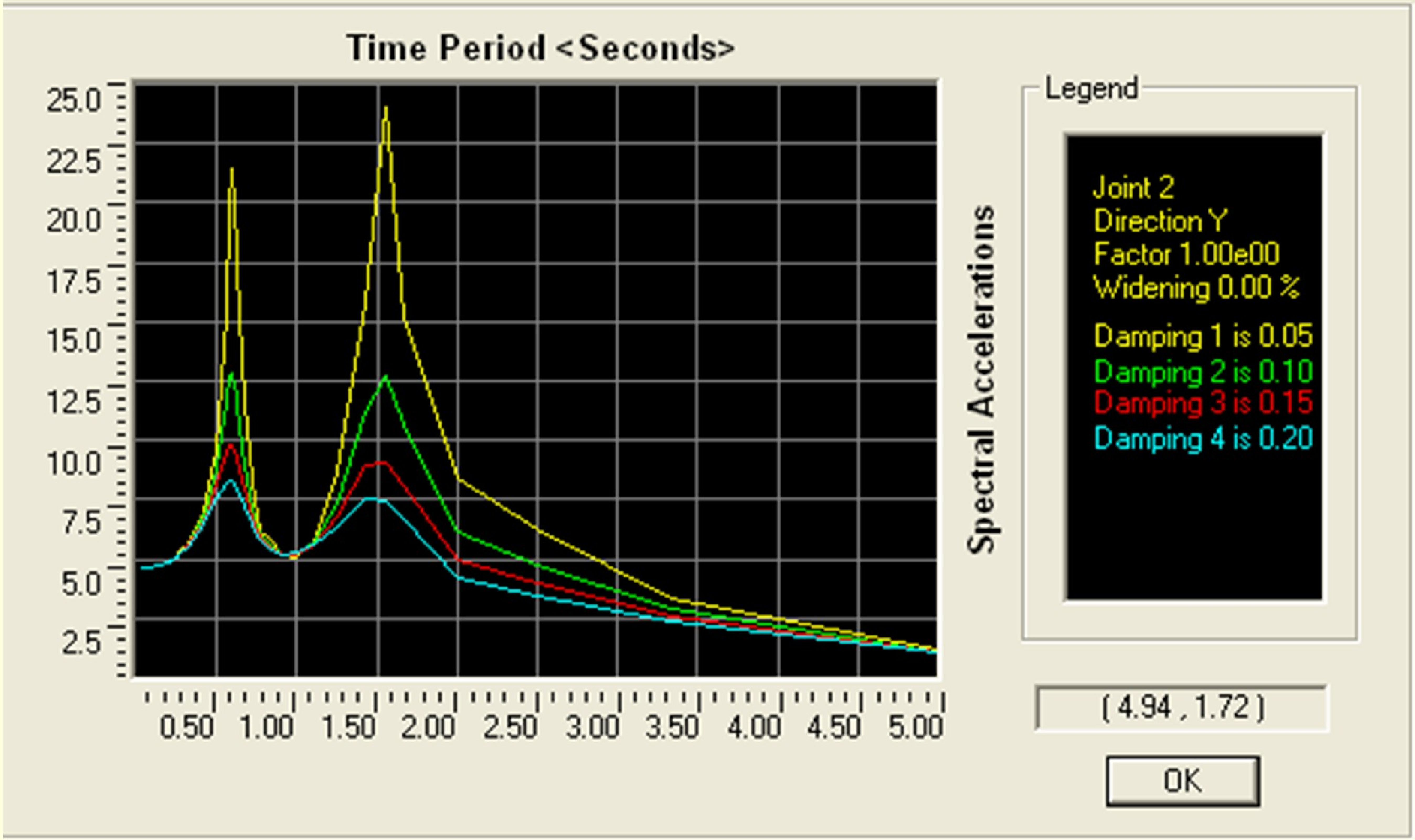
$$M_T = (1.17)^2 268 + (-0.17)^2 703 = 389$$

$$MMPR = \left[\Gamma_n{}^2 M_n / M_T \right] = \begin{bmatrix} 0.95 \\ 0.05 \end{bmatrix}$$

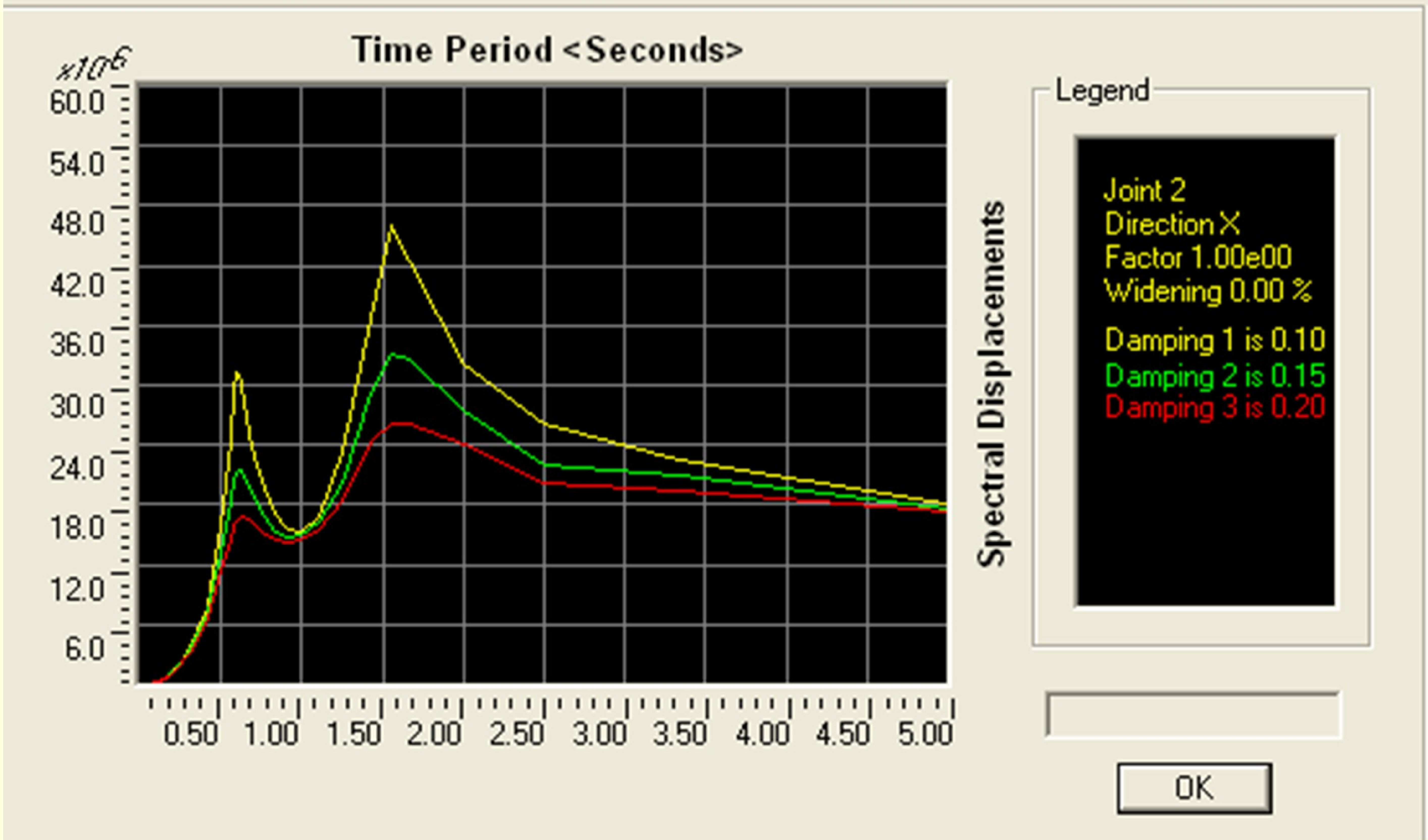
Response spectrum for x-accelration (elcentro)



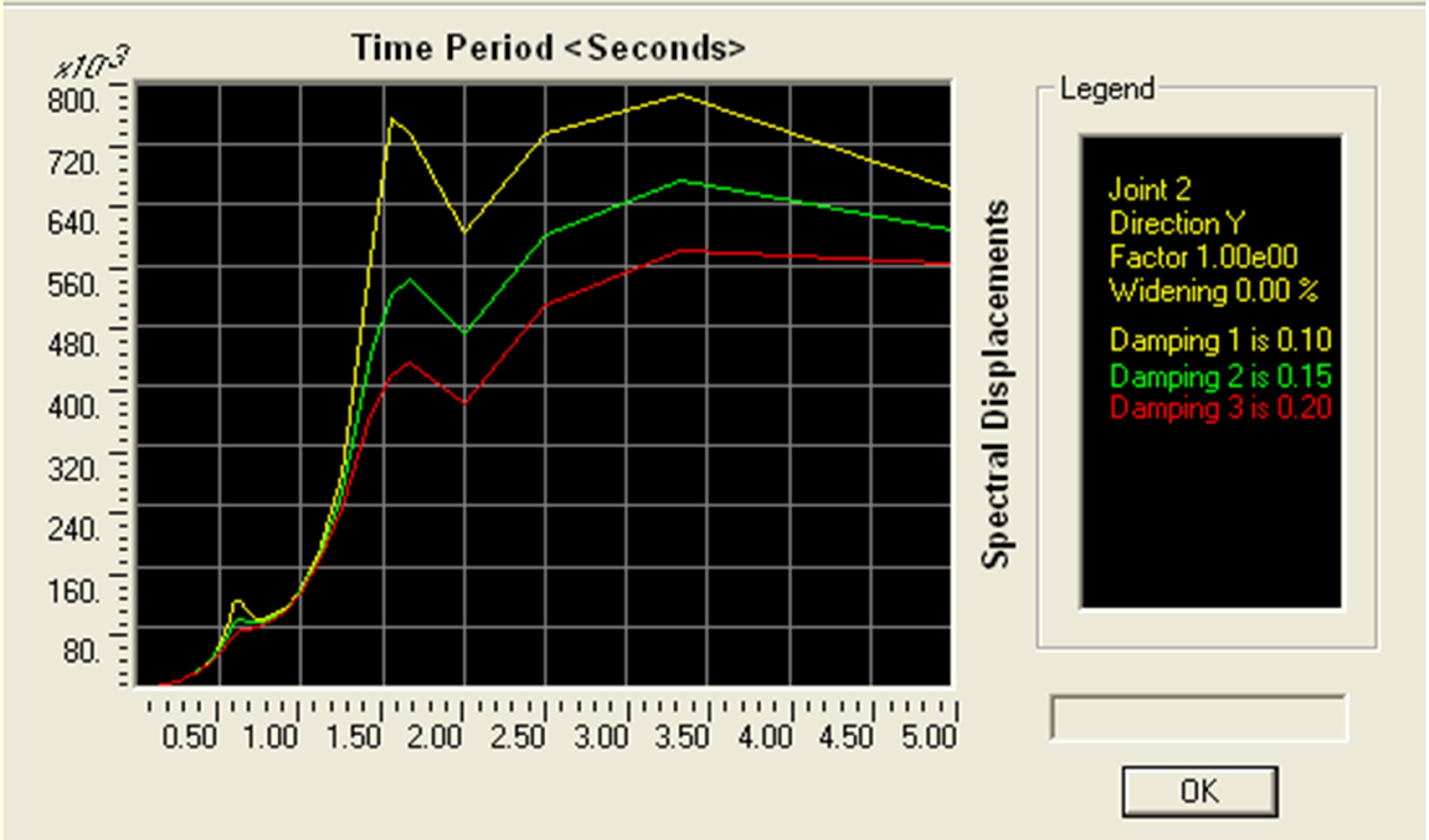
Response spectrum for y-accelration (elcentro)



Response spectrum for x-displacement (elcentro)



Response spectrum for y-displacement (elcentro)

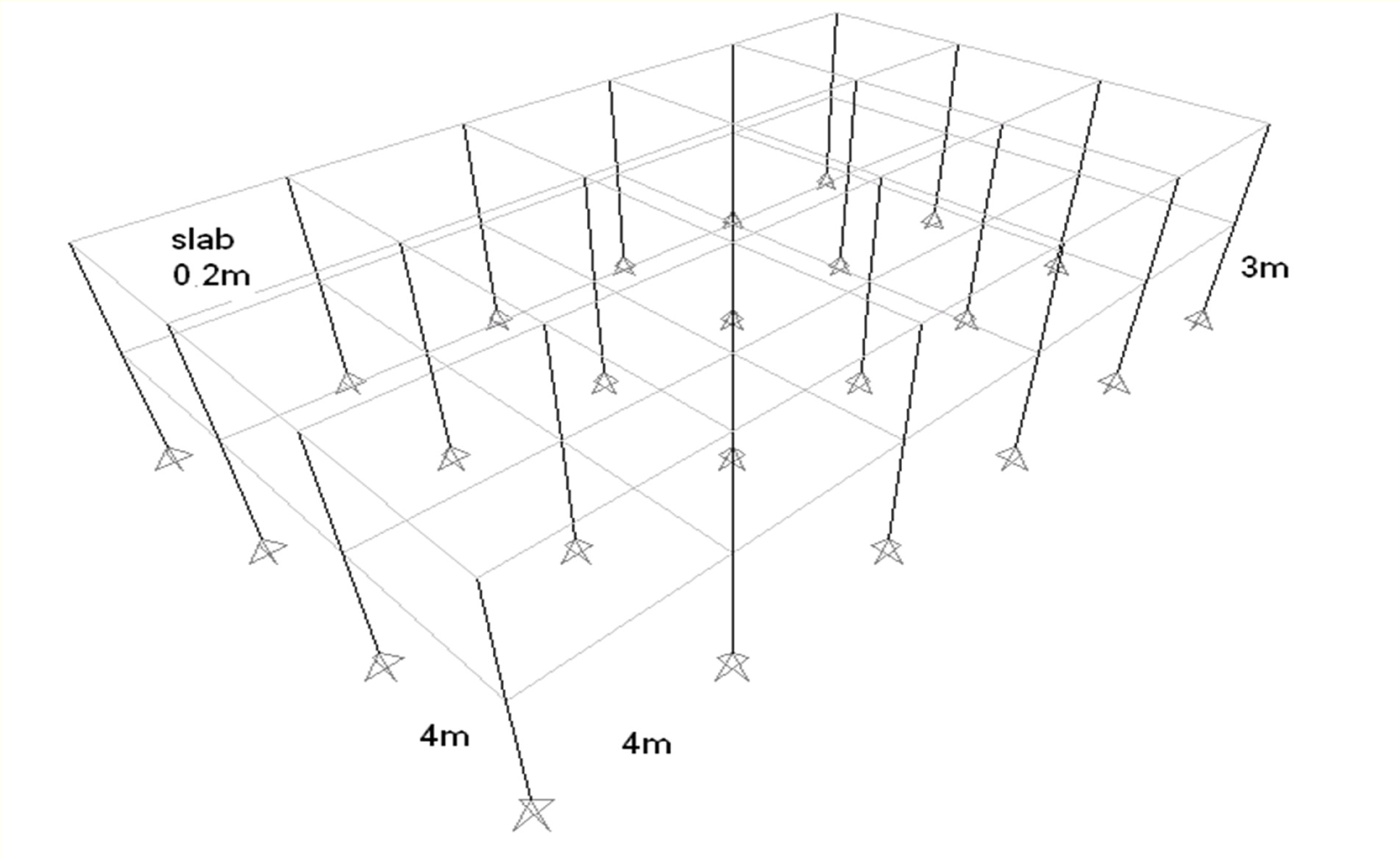


Analysis of response spectrum curves

- At short periods the structure is rigid and has the same acceleration of the ground. The total motion is the same as the ground motion and the relative motion is almost zero.
- At long periods the structure is flexible and has almost zero acceleration. The total motion is almost zero and the relative motion is equal to ground motion.
- Between the previous two boundaries, the acceleration of the structure is very large at periods of forcing function that are equal or close to the natural structure periods. And since there are many of them the design of structure in this range is highly vulnerable.

Homework #6

- RC flat plate structure shown next page
- Square columns 30cm
- No superimposed loads
- $E=25GPa$, $\mu=0.2$, $\rho=2.5t/m^3$
- Find acceleration of first and second floor if:
- 1. structure is subjected to earthquake acceleration 0.3g in x-direction:
  - Uniform for 1-sec
  - Sinusoidal with period: 0.001, 0.148, 0.3, 0.67, 6.7sec
- 2. Structure is subjected to Elcentro earthquake


slab
0.2m
3m
4m
4m

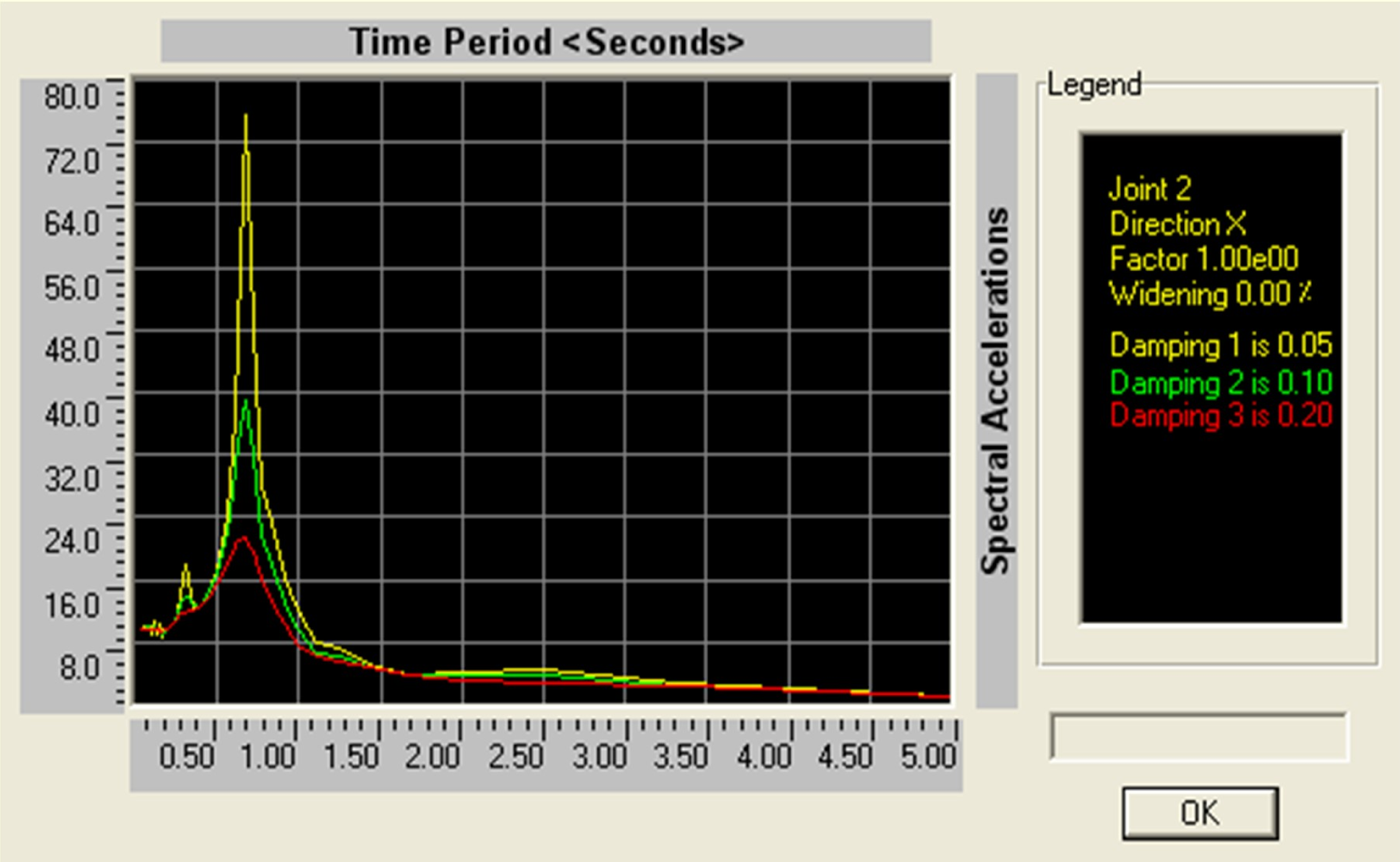
Modal Analysis

participation direction	Modal mass particip. ratio	period	Mode number
Uy	0.98	0.671	1
Ux	0.98	0.671	2
Rz	0.97	0.605	3
Uy, Ux	0.02	0.148	4,5

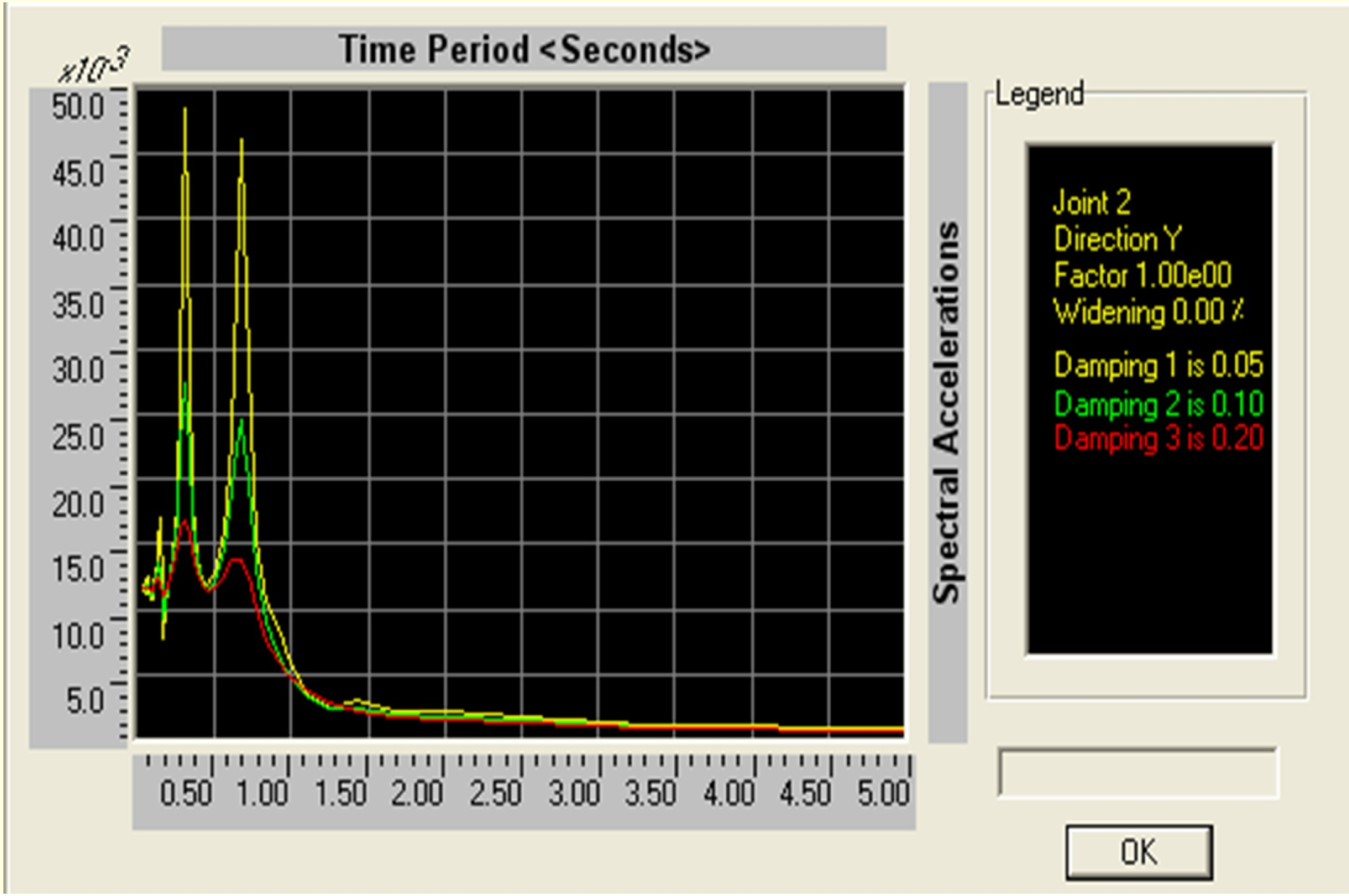
SAP output/ to be continued

Displ. 2nd m	Displ. 1st m	Accel. 2nd m/s^2	Accel. 1st m/s^2	
0.076	0.056	3.70	2.94	Uniform
0	0	2.94	2.94	Sin0.001
Increase With time	Increase With time	Increase With time	Increase With time	Sin0.148,sin0.67
0.031	0.022	6.2	4.4	Sin0.3
0.041	0.031	0.26	0.2	Sin6.7
0.084	0.062	8.9	7.3	elcentro

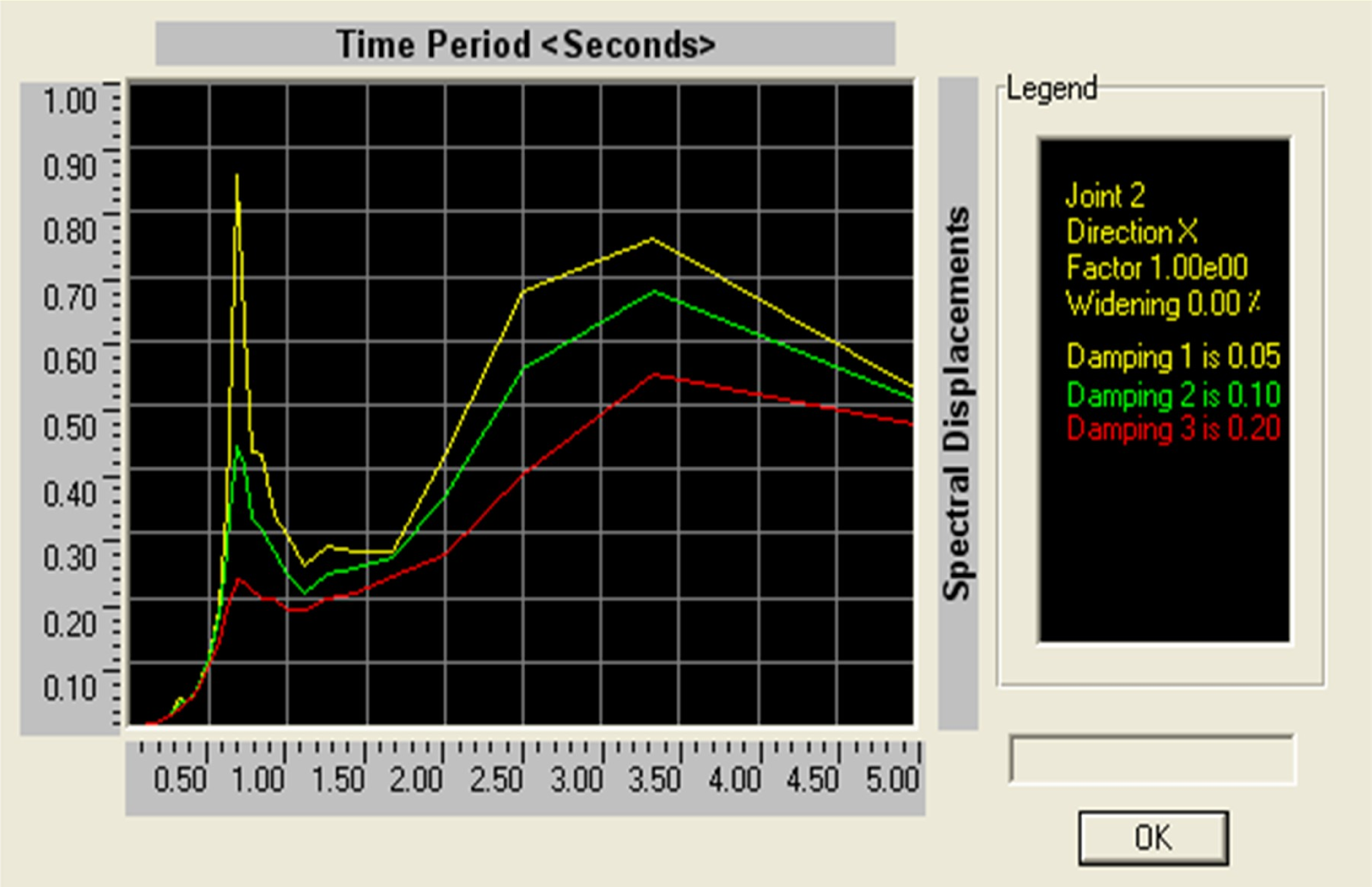
Response spectrum for x-acceleration



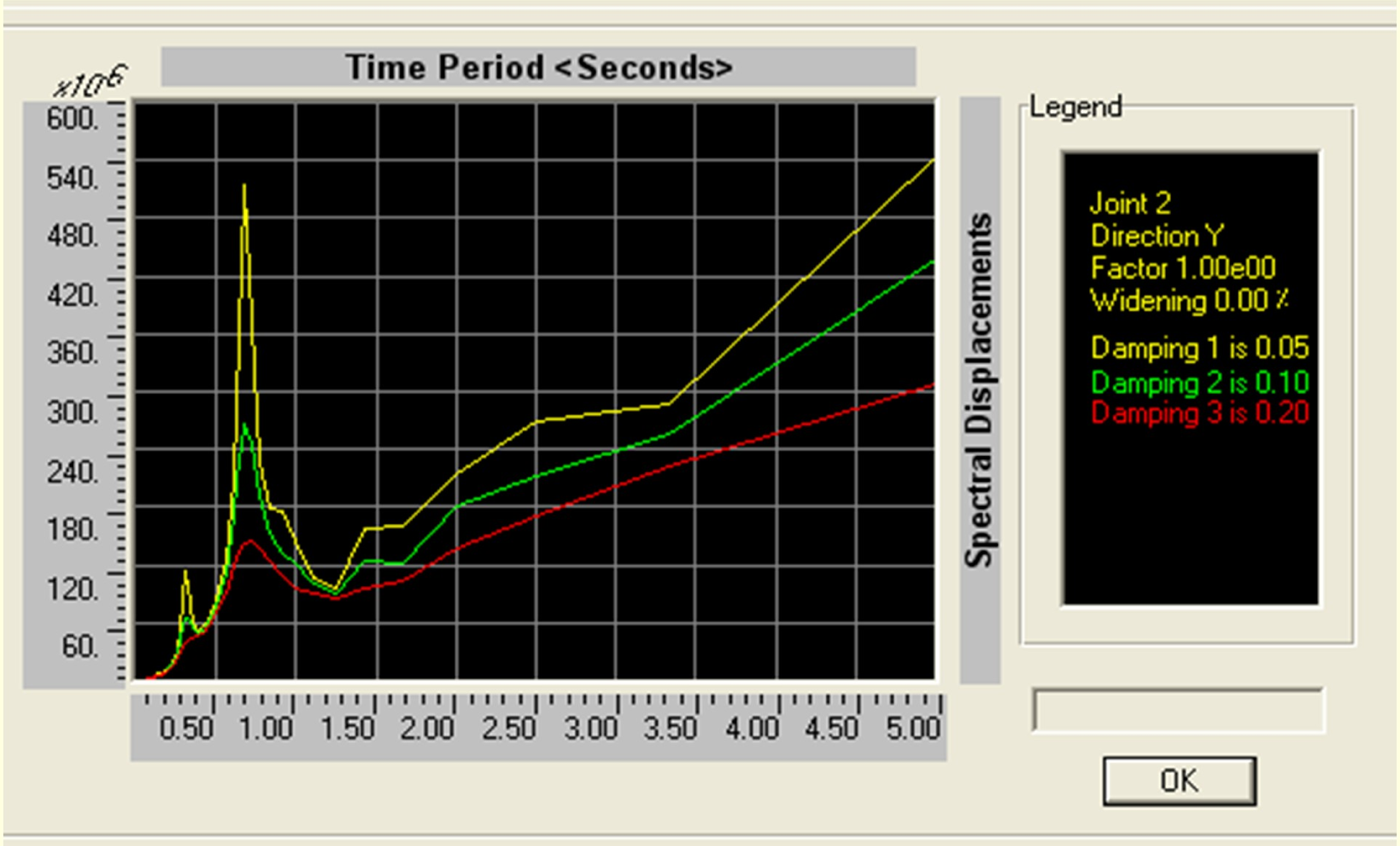
Response spectrum for y-acceleration



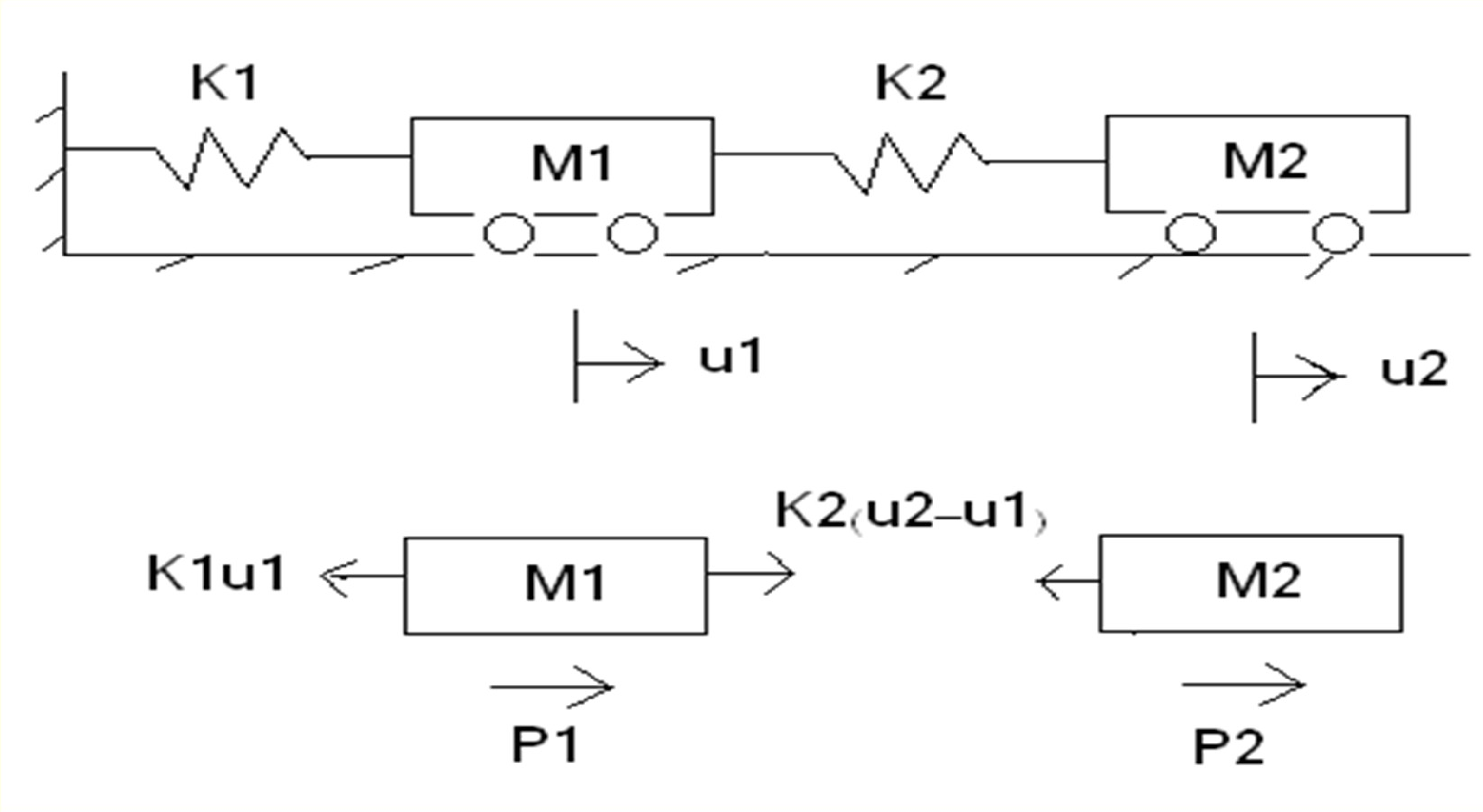
Response spectrum for x-displacement



Response spectrum for y-displacement



Undamped Forced Vibrations
Analytical solution for analogical model



Equations of motion for uniform acceleration

$$M\ddot{u}^{*} + Ku^{*} = -Ma_g$$

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, K = \begin{pmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{pmatrix}$$

$$P = \begin{pmatrix} -M_1 \\ -M_2 \end{pmatrix} a_g$$

Solution of set of differential equations:
Particular solution:

$$u_p = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, ([K])\{u_p\} = \{c\}$$

$$\{u_p\} = ([K])^{-1}\{c\}$$

$$\{u\} = \{u_h\} + \{u_p\}$$

General Solution of 2-stories

$$u_1 = r_1(a_1 \cos\omega_1 t + a_2 \sin\omega_1 t) +$$
$$r_2(a_3 \cos\omega_2 t + a_4 \sin\omega_2 t) + c_1$$

$$u_2 = (a_1 \cos\omega_1 t + a_2 \sin\omega_1 t) +$$
$$(a_3 \cos\omega_2 t + a_4 \sin\omega_2 t) + c_2$$

Example solution: M=194.4t, K=8888kN/m

$$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} + \begin{pmatrix} 2K & -K \\ -K & K \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} M(0.3g)$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{K^2}\begin{pmatrix} K & K \\ K & 2K \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} M(0.3g) = \begin{pmatrix} -2 \\ -3 \end{pmatrix}\frac{(0.3g)M}{K}$$

$$u_1 = r_1(a_1 \cos\omega_1 t + a_2 \sin\omega_1 t) +$$

$$r_2(a_3 \cos\omega_2 t + a_4 \sin\omega_2 t) - \frac{0.6Mg}{K}$$

$$u_2 = (a_1 \cos\omega_1 t + a_2 \sin\omega_1 t) +$$

$$(a_3 \cos\omega_2 t + a_4 \sin\omega_2 t) - \frac{0.9Mg}{K}$$

Example solution cont

$$x=0: u_1=0=r_1a_1+r_2a_3-\frac{0.6Mg}{K}, u_2=0=a_1+a_3-\frac{0.9Mg}{K}$$

$$\begin{pmatrix} r_1 & r_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_3 \end{pmatrix}=\begin{pmatrix} 0.6 \\ 0.9 \end{pmatrix}\frac{Mg}{K}$$

$$\begin{pmatrix} a_1 \\ a_3 \end{pmatrix}=\frac{1}{r_1-r_2}\begin{pmatrix} 1 & -r_2 \\ -1 & r_1 \end{pmatrix}\begin{pmatrix} 0.6 \\ 0.9 \end{pmatrix}\frac{Mg}{K}=\begin{pmatrix} 0.92 \\ -0.02 \end{pmatrix}\frac{Mg}{K}$$

$$x=0: \dot{u}_1=0=r_1a_2\omega_1+r_2a_4\omega_2, \dot{u}_2=0=a_2\omega_1+a_4\omega_2, a_2=a_4=0$$

Example solution cont

$$u_1 = (0.618(.92\cos\omega_1 t) - 1.618(-0.02\cos\omega_2 t) - 0.6)\frac{Mg}{K}$$

$$u_1 = 0.12\cos\omega_1 t + 0.006\cos\omega_2 t - 0.13$$

$$u_2 = (.92\cos\omega_1 t - 0.02\cos\omega_2 t - 0.9)\frac{Mg}{K}$$

$$u_2 = 0.2\cos\omega_1 t - 0.004\cos\omega_2 t - 0.193$$

Comparison between static and dynamic analysis

- From

$$\begin{pmatrix} 2K & -K \\ -K & K \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} M(0.3g)$$

- Insert K=8888, M=194.4 to get: u_1=.13m, u_2=.192m
- Maximum response expected adding amplitudes of previous slide u_1 =0.26 (SAP=0.28m), u_2=0.40m (SAP 0.42m)

Equations of motion for cos/sin acceleration

$$M\ddot{u}^* + Ku^* = -Ma_g \cos \varpi t$$

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, K = \begin{pmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{pmatrix}$$

$$P = \begin{pmatrix} -M_1 \\ -M_2 \end{pmatrix} a \cos \varpi t$$

Solution of set of differential equations

Particular solution:

$$u^{*} = \begin{pmatrix} u^{*}{}_{1} \\ u^{*}{}_{2} \end{pmatrix} \cos \varpi t$$

$$\left([K] - \varpi^{2} [M] \right) \{u^{*}\} = \{P\}$$

$$\{u^{*}\} = \left([K] - \varpi^{2} [M] \right)^{-1} \{P\}$$

Comments:

1. Problem of finding inverse with increasing number of degrees of freedom.
2. For slowly varying disturbing force (i.e. ϖ approaches zero)

$$\{u^*\} = \left([K] - \varpi^2 [M]\right)^{-1} \{P\} = \left([K]\right)^{-1} \{P\}$$

Static solution

3.Notice displacement goes to infinity when ϖ equal to either ω_1 or ω_2

Damping

1. Structural damping is not viscous.
2. Due to mechanisms such as hysteresis and slip in connections.
3. Mechanisms not well understood.
4. Awkward to incorporate into structural dynamic equations.
5. Makes equations computationally difficult.
6. Effects usually approximated by viscous damping.

Critical Damping

$$\xi \qquad \textit{Fraction of Critical Damping}$$

$$\xi = 1 \qquad \textit{Critical Damping}$$

Critical Damping marks the transition between oscillatory and non- oscillatory response of a structure

Critical Damping Ratio

$0.5\% \leq \xi \leq 5\%$	*Steel Piping*
$2\% \leq \xi \leq 15\%$	*Bolted or riveted steel structures*
$2\% \leq \xi \leq 15\%$	*Reinforced or Prestresses Concrete*

Actual value may depend on stress level.

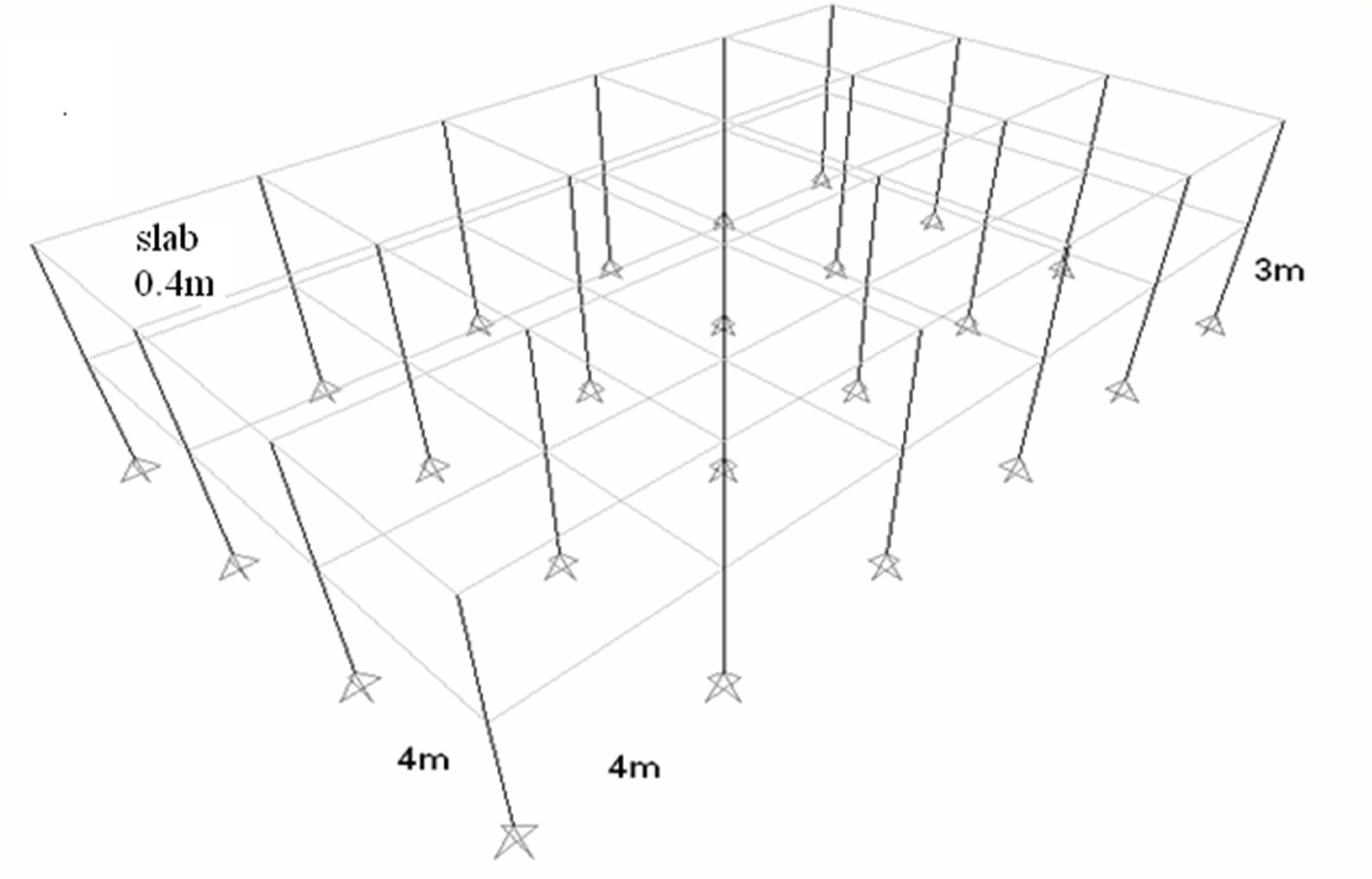
Rayleigh or Proportional Damping

Damping matrix is a linear combination of stiffness and mass matrices:

$$[C] = \alpha [K] + \beta [M]$$

Homework #7

- RC flat plate structure shown next page
- Square columns 20cm
- No superimposed loads
- $E=25GPa$, $\mu=0.2$, $\rho=2.5t/m^3$
- Find acceleration of first and second floor if:
- 1. structure is subjected to earthquake acceleration 0.3g in x-direction:
  - Uniform for 6-sec
  - Sinusoidal with period: 0.001, 0.15, 0.35, 0.65, 1.5, 6 sec
- 2. Structure is subjected to Elcentro earthquake


slab
0.4m
3m
4m
4m

Modal Analysis: one story

participation direction	Modal mass particip. ratio	period	Mode number
Uy	1	1.05	1
Ux	1	1.05	2
Rz	1	0.875	3
Uz	0.846	0.047	4

Homework # 7 Modeling as SDOF

$$M = (12*20*.4 + 24*1.5*0.2*.2)*2.5 = 243.6t$$

$$K_X = K_Y = 24*\frac{3*EI}{L^3} = 8888KN/m$$

$$T_X = T_Y = 2\pi\sqrt{M/K} = 1.04\sec$$

- Homework: do analogical solutions for modes 2-4

Modal Analysis: two stories hw: verify manually

participation direction	Modal mass particip. ratio	period	Mode number
Uy	0.995	1.55	1
Uy	0.005	0.364	4
participation direction	Modal Amplitude	Modal Amplitude	Mode number
Uy	r_1=0.87	r_2=-1.14	1,4

Hw#7 cont

$$u_1 = (0.6\cos\omega_1 t + 0.002\cos\omega_2 t - 0.6)\frac{Mg}{K}$$

$$u_1 = 0.16\cos\omega_1 t + 0.0005\cos\omega_2 t - 0.16$$

$$u_2 = (.679\cos\omega_1 t - 0.002\cos\omega_2 t - 0.675)\frac{Mg}{K}$$

$$u_2 = 0.18\cos\omega_1 t - 0.0005\cos\omega_2 t - 0.18$$

Comparison between static and dynamic analysis

- From

$$\begin{pmatrix} 5K & -4K \\ -4K & 4K \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} M(0.3g)$$

- Insert K=8888, M=243.6 to get: u_1=.16m, u_2=.18m
- Maximum response expected adding amplitudes of previous slide u_1 =0.32 (SAP=0.33m), u_2=0.36m (SAP 0.38m)

Normal Mode response to applied actions

When [K], [M] are known and time independent the problem is linear.

$$[M]\{\ddot{u}\} + [K]\{u\} = \{A\}$$

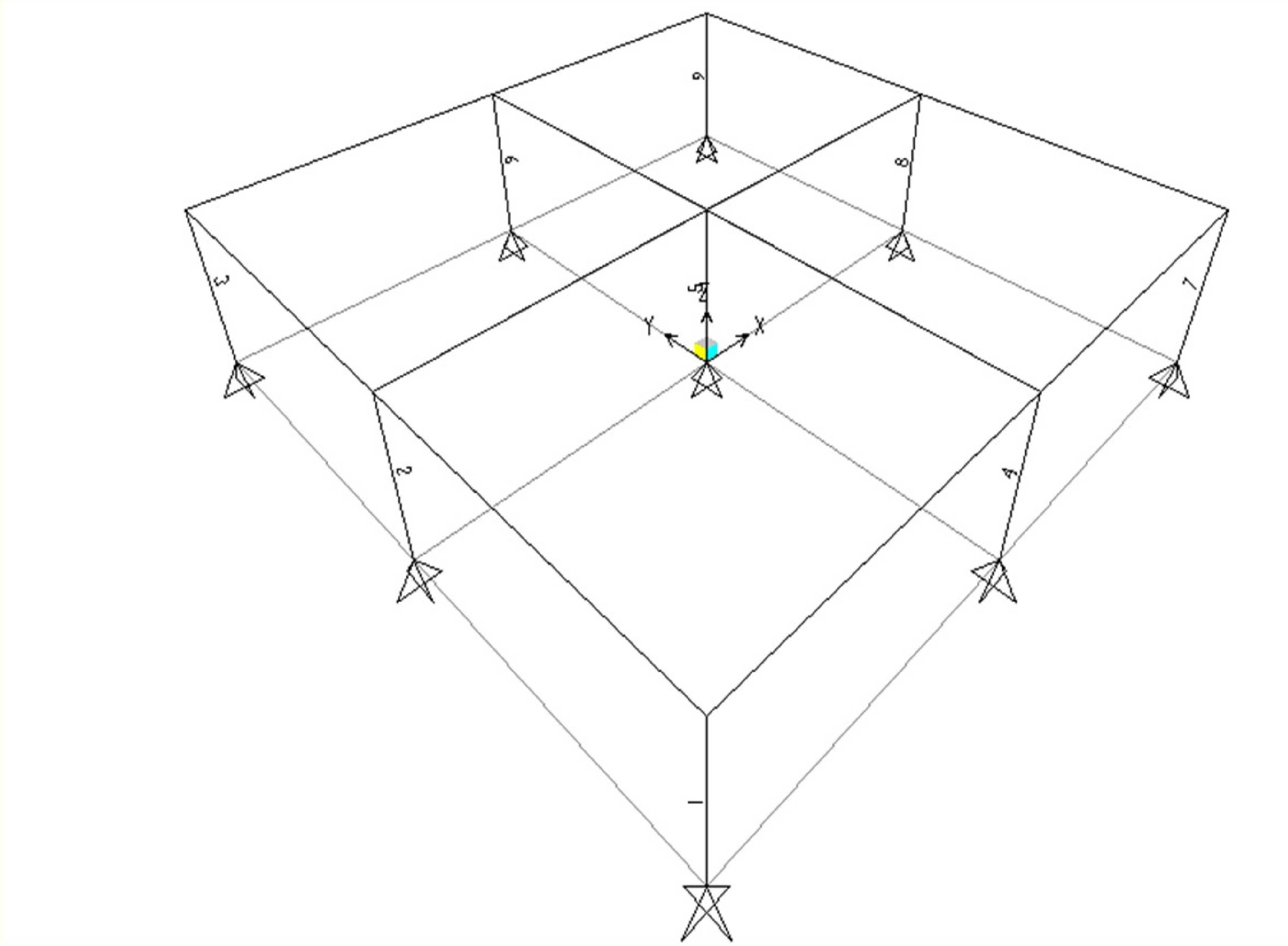
$$\varphi^T [M] \varphi\varphi^{-1} \{\ddot{u}\} + \varphi^T [K] \varphi\varphi^{-1} \{u\} = \varphi^T \{A\}$$

$$[M_P] \varphi^{-1} \{\ddot{u}\} + [K_P] \varphi^{-1} \{u\} = \varphi^T \{A\}$$

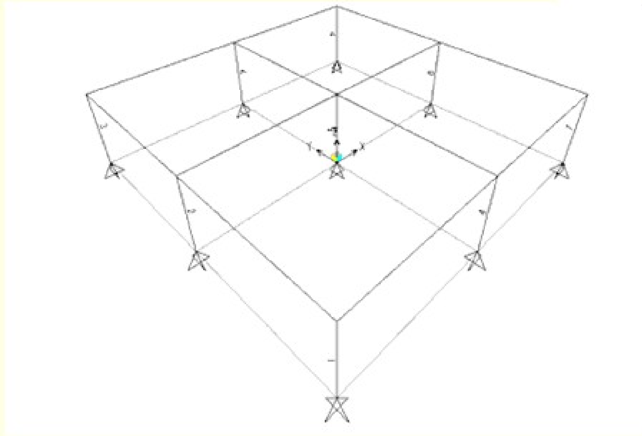
$$[M_P]\{\ddot{u}_P\} + [K_P]\{u_P\} = \{A_P\}$$

Parameters affecting period calculations

Period changes with ratio of panels, story height, …etc. Only factors related to concrete strength, superimposed loads and connection between structure and base will be discussed in the next example.



a one-story flat plate reinforced concrete building subjected to earthquake loading in the x-direction. The thickness of slab is 20cm, columns are square having 40cm dimensions, 3m elevation and the distance between them is 6m.



- Based on material properties, superimposed loads and boundary conditions three cases are considered.
- basic case:
  - the concrete cube strength = 25000KN/m^2 (Ecol=20GN/m^2)
  - the superimposed dead load = 300kg/m^2
  - the connection of the building to the ground is pinned

  T= 0.62sec
- first case: the concrete cube strength = 50000KN/m^2 (Ecol=30GN/m^2): T= 0.51sec
- second case: the superimposed dead load = 100kg/m^2

  T=0.44sec
- Third case: the connection of the building to the ground is fixed

  T=0.19sec
- IBC2003: $T=0.073H_N^{3/4}$ (H in meters)=

End of dynamic analysis, part 3

Let Learning Continue