Chapter 4

The wave like properties of particle
Louis de Broglie

- 1892 – 1987
- French physicist
- Originally studied history
- Was awarded the Nobel Prize in 1929 for his prediction of the wave nature of electrons
Wave Properties of Particles

• de Broglie postulated that because photons have both wave and particle characteristics, perhaps all forms of matter have both properties

• The de Broglie wavelength of a particle is

\[ \lambda = \frac{h}{p} = \frac{h}{mu} \]
Frequency of a Particle

• In an analogy with photons, de Broglie postulated that a particle would also have a frequency associated with it

\[ \nu = f = \frac{E}{h} \]

• These equations present the **dual nature** of matter
  – Particle nature, \( p \) and \( E \)
  – Wave nature, \( \lambda \) and \( f = \nu \)
Davisson-Germer Experiment

- If particles have a wave nature, then under the correct conditions, they should exhibit diffraction effects
- Davisson and Germer measured the wavelength of electrons
- This provided experimental confirmation of the matter waves proposed by de Broglie
Complementarity

• The **principle of complementarity** states that the wave and particle models of either matter or radiation complement each other.

• **Neither** model can be used exclusively to describe matter or radiation adequately.
Electron Diffraction, Experiment

• Parallel beams of mono-energetic electrons that are incident on a double slit
• The slit widths are small compared to the electron wavelength
• An electron detector is positioned far from the slits at a distance much greater than the slit separation
Electron Diffraction, Set-Up
Electron Diffraction, cont.

- If the detector collects electrons for a long enough time, a typical wave interference pattern is produced.
- This is distinct evidence that electrons are interfering, a wave-like behavior.
- The interference pattern becomes clearer as the number of electrons reaching the screen increase.
Electron Diffraction, Equations

• A maximum occurs when $d \sin \theta = n\lambda$
  – This is the same equation that was used for light

• This shows the dual nature of the electron
  – The electrons are detected as particles at a localized spot at some instant of time
  – The probability of arrival at that spot is determined by finding the intensity of two interfering waves
Electron Diffraction Explained

• An electron interacts with both slits simultaneously

• If an attempt is made to determine experimentally which slit the electron goes through, the act of measuring destroys the interference pattern
  – It is impossible to determine which slit the electron goes through

• In effect, the electron goes through both slits
  – The wave components of the electron are present at both slits at the same time
Electron Microscope

• The electron microscope relies on the wave characteristics of electrons
• The electron microscope has a high resolving power because it has a very short wavelength
• Typically, the wavelengths of the electrons are about 100 times shorter than that of visible light
Wave

- Three kinds of wave in Nature: mechanical, electromagnetically and matter waves
- The simplest type of wave is strictly sinusoidal and is characterised by a `sharp’ frequency
- \( v = 1/T, \ T = \) the period of the wave
- wavelength \( \lambda \) and its travelling at speed \( = c \)

\[
y = A \cos \left( kx - \omega t \right)
\]

\[
c = \lambda v; \ k = \frac{2\pi}{\lambda}
\]

A `pure’ (or ‘plain’) wave which has `sharp’ wavelength and frequency
Where is the wave?

- For the case of a **particle** we can locate its **location** and **momentum** precisely.
- But how do we ‘locate’ a wave?

Wave spreads **out** in a region of space and is **not** located in any **specific point** in space like the case of a particle.

- To be more precise we say that a **plain** wave exists **within** some region in space, $\Delta x$.
- For a particle, $\Delta x$ is just the ‘**size**’ of its dimension. In principle, we can determine the position of $x$ to infinity.
- But for a **wave**, $\Delta x$ could be **infinity**.
A pure wave has $\Delta x \to \infty$

- If we know the wavelength and frequency of a pure wave with **infinite precision**
- the statement that the wave number and frequency are ‘sharp’, one can shows that :
- The wave **cannot** be confined to any **restricted** region of space but must have an **infinite extension** along the direction in which it is propagates

- In other words, the wave is ‘**everywhere**’ when its wavelength is ‘**sharp**’
- This is what it means by the mathematical statement that “
  - $\Delta x$ is infinity”
The Uncertainty Relationships (UR)

• In classical mechanics, it is possible, in principle, to make measurements with arbitrarily small uncertainty.

• The relationships for classical waves are:
  \[ \Delta x \Delta k \approx 1 \]  
  is the wave number – position (UR)
  \[ \Delta w \Delta t \approx 1 \]  
  is the frequency – time (UR)
But the wave number is 
k = \frac{2\pi}{\lambda}

Then \( \Delta k = - \left( \frac{2\pi}{\lambda^2} \right) \Delta \lambda \)

Use this in the equation \( \Delta x \Delta k \), we get

\[ \Delta x \Delta \lambda = - \left( \frac{\lambda^2}{2\pi} \right) \text{ is wavelength- position (UR)} \]

\[ \approx \left( \frac{\lambda^2}{2\pi} \right) \]
• $\Delta x \Delta \lambda \geq \lambda^2$ can also be expressed in an equivalence form:

$$\Delta t \Delta \nu \geq 1$$

via the relationship $c = \nu \lambda$ and $\Delta x = c \Delta t$

• Where $\Delta t$ is the **time required** to measure the frequency of the wave.

• The more we know about the value of the **frequency** of the wave, the **longer** the time taken to measure it.

• If you want to know **exactly** the precise value of the frequency, the required time is $\Delta t = \textit{infinity}$.
Quantum theory

Quantum theory predicts that it is fundamentally impossible to make simultaneous measurements of a particle’s position and momentum with infinite accuracy.
Werner Heisenberg

• 1901 – 1976
• German physicist
• Developed *matrix mechanics*
• Many contributions include:
  – Uncertainty principle
    • Rec’d Nobel Prize in 1932
  – Prediction of two forms of molecular hydrogen
  – Theoretical models of the nucleus
Heisenberg Uncertainty Relationships, Statement (HUR)

• The **Heisenberg uncertainty principle** states: if a measurement of the **position** of a particle is made with **uncertainty** \( \Delta x \) and a simultaneous measurement of its x component of **momentum** is made with **uncertainty** \( \Delta p_x \), the product of the **two uncertainties** can never be **smaller** than \( \frac{\hbar}{2} \), where \( \hbar = 2\pi\hbar \)

\[
\Delta x \Delta p_x \geq \frac{\hbar}{2}
\]
Heisenberg Uncertainty Principle, Explained

• It is physically impossible to measure **simultaneously** the **exact** position and **exact** momentum of a particle

• The inescapable (unavoidable) uncertainties do not arise from imperfections (The state or an instance of being imperfect) in practical measuring instruments

• The **uncertainties** arise from the **quantum** structure of matter
Heisenberg Uncertainty Principle, Another Form

• Another form of the uncertainty principle can be expressed in terms of energy and time

\[ \Delta E \Delta t \geq \frac{\hbar}{2} \]

• This suggests that energy conservation can appear to be violated by an amount \( \Delta E \) as long as it is only for a short time interval \( \Delta t \)
Quantum Particle

• The quantum particle is a new model that is a result of the recognition of the dual nature
• Entities have both particle and wave characteristics
• We must choose one appropriate behavior in order to understand a particular phenomenon
Ideal Particle vs. Ideal Wave

• An ideal particle has zero size
  – Therefore, it is localized in space
• An ideal wave has a single frequency and is infinitely long
  – Therefore, it is unlocalized in space
• A localized entity can be built from infinitely long waves
Particle as a Wave Packet

- Multiple waves are superimposed so that one of its crests is at $x = 0$
- The result is that all the waves add constructively at $x = 0$
- There is destructive interference at every point except $x = 0$
- The small region of constructive interference is called a wave packet
  - The wave packet can be identified as a particle
Wave packets (wp)

• is superposition (addition) of large number of waves, which interfere **constructively** in the vicinity of the particle, giving the resultant wave with a **large** amplitude and interfere **destructively** far from the particle, so that the resultant wave has a **small** amplitude in regions where we **don’t** expect to find the particle.

• For example, an electron is bound to a specify atom. Its position is known **within uncertainty** of the order of the **diameter** of the atom (10^{-10}) meter, but we **don’t** know exactly where it is within the atom. So **wave packet** method is used to describe such situation.
Wave Envelope

- The blue line represents the **envelope** function.
- This envelope can travel through space with a **different speed** than the individual waves.
Definition of the phase and the group velocities

• The phase velocity:
  is the velocity at which individual component wave move within the envelope.

The group velocity:
  is the velocity at which the envelope of the wave packet moves.

• They are given by the following relations:
Speeds Associated with Wave Packet

• The **phase speed** of a wave in a wave packet is given by

\[ v_{phase} = \frac{\omega}{k} \]

– This is the rate of advance of a crest on a single wave

• The group speed is given by

\[ v_g = \frac{d\omega}{dk} \]

– This is the speed of the wave packet itself
The group speed can also be expressed in terms of energy and momentum:

\[
\nu_g = \frac{dE}{dp} = \frac{d}{dp} \left( \frac{p^2}{2m} \right) = \frac{1}{2m} (2p) = u
\]

This indicates that the group speed of the wave packet is identical to the speed of the particle \( u \) that it is modeled to represent.
More quantitatively, 
\[ \Delta x \Delta \lambda \geq \lambda^2 \]

- This is the uncertainty relationships for classical waves 
  \( \Delta \lambda \) is the uncertainty in the wavelength.

- When the wavelength `sharp` (that we knows its value precisely), this would mean \( \Delta \lambda = 0 \).

- In other words, \( \Delta \lambda \to \) infinity means we are totally ignorant of what the value of the wavelength of the wave is.
Wave can be made more ``localized’’

• We have already shown that the 1-D plain wave is **infinite** in extent and **can’t** be properly localized

  (because for this wave, $\Delta x \rightarrow \text{infinity}$)

• However, we can construct a relatively localized wave (i.e., with **smaller** $\Delta x$) by :

  • **adding** up two plain waves of slightly different wavelengths (or equivalently, frequencies)
\( \Delta x \) is the uncertainty in the location of the wave (or equivalently, the region where the wave exists)

- \( \Delta x = 0 \) means that we know exactly where the wave is located,

- where as \( \Delta \lambda \to \infty \) means the wave is spread to all the region and we cannot tell where is it’s `location’

The more we know about \( x \), the less we know about \( \lambda \) as \( \Delta x \) is inversely proportional to \( \Delta \lambda \)
Constructing wave groups

- Two pure waves with slight **difference** in frequency and wave number $\Delta w = w_1 - w_2$, $\Delta k = k_1 - k_2$, are superimposed

\[ y_1 = A \cos(k_1x - \omega_1t); \quad y_2 = A \cos(k_2x - \omega_2t) \]
Envelop wave and phase wave

The resultant wave is a ‘wave group’ comprise of an ‘envelop’ (or the group wave) and a phase waves

\[ y = y_1 + y_2 \]

\[ = 2A \cos \frac{1}{2} \left( \{k_2 - k_1\} x - \{\omega_2 - \omega_1\} t \right) \cdot \cos \left( \frac{k_2 + k_1}{2} \right) x - \left( \frac{\omega_2 + \omega_1}{2} \right) t \]
• As a comparison to a plain waves, a group wave is more ‘localized’ (due to the existence of the wave envelop. In comparison, a plain wave has no `envelop' but only `phase wave’)

• It comprises of the slow envelop wave

$$2A \cos \frac{1}{2} (\{k_2 - k_1\}x - \{\omega_2 - \omega_1\}t) = 2A \cos \frac{1}{2} (\Delta k x - \Delta \omega t)$$

that moves at group velocity $v_g = \Delta \omega / \Delta k$

and the phase waves (individual waves oscillating inside the envelop)

$$\cos \left\{ \left( \frac{k_2 + k_1}{2} \right)x - \left( \frac{\omega_2 + \omega_1}{2} \right)t \right\} = \cos \{k_p x - \omega_p t\}$$

moving at phase velocity $v_p = \omega_p / k_p$

In general, $v_g = \Delta \omega / \Delta k << v_p = (\omega_1 + \omega_2)/(k_1 + k_2)$ because $\omega_2 \approx \omega_1, \quad k_1 \approx k_2$
\[ y = y_1 + y_2 = \left\{ 2A \cos \frac{1}{2} (\Delta k x - \Delta \omega t) \right\} \cdot \cos \{ k_p x - \omega_p t \} \]

‘envelop’ (group waves).
Sometimes it’s called
‘modulation’
Why are waves and particles so important in physics?

- **Waves and particles** are important in physics because they represent the only modes of energy transport (interaction) between two points.
Energy is carried at the speed of the group wave

- The energy carried by the group wave is concentrated in regions in which the amplitude of the envelope is large.
- The speed with which the waves' energy is transported through the medium is the speed with which the envelope advances, not the phase wave.
- In this sense, the envelop wave is of more ‘physical’ relevance in comparison to the individual phase waves (as far as energy transportation is concerned).
Wave pulse – an even more `localized’ wave

- In the previous example, we add up only two slightly different wave to form a train of wave group.

- An even more `localized’ group wave – what we call a “wave pulse” can be constructed by adding more sine waves of different numbers \( k_i \) and possibly different amplitudes so that they interfere constructively over a small region \( \Delta x \) and outside this region they interfere destructively so that the resultant field approach zero.

Mathematically,

\[
y_{\text{wave pulse}} = \sum_{i}^{\infty} A_i \cos (k_i x - \omega_i t)
\]
A wave pulse – the wave is well localized within $\Delta x$. This is done by adding a lot of waves with their wave parameters $\{A_i, k_i, w_i\}$ slightly differ from each other ($i = 1, 2, 3\ldots$ as many as it can)

such a wave pulse will move with a velocity

$$v_g = \frac{d\omega}{dk} \bigg|_{k_0}$$

( the group velocity considered earlier $v_g = \Delta \omega / \Delta k$ )
Comparing the three kinds of wave

Which wave is the most localized?
Waves superimpose, not collide

• In contrast, two waves do not interact in the manner as particle-particle or particle-wave do.

• Wave and wave simply “superimpose”: they pass through each other essentially unchanged, and their respective effects at every point in space simply add together according to the principle of superposition to form a resultant at that point -- a sharp contrast with that of two small, impenetrable particles.
Superposition of waves

(a) $y_1$  $y_2$

(b) $y_1 + y_2$

(c) $y_1 + y_2$

(d) $y_2$  $y_1$
Assignments on ch 4

- # 4
- 1.6.15
- #
- 18.22.27