Coherent phase shift keying

- In coherent phase shift keying different phase modulation schemes will be covered i.e. binary PSK, quadrature phase shift keying and M-ary PSK

Binary Phase shift keying

- In a coherent PSK system the pair of signals $s_1(t)$ and $s_2(t)$ are used to represent binary logics 1 and 0 respectively

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t)$$

$$s_2(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t + \pi) = -\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t)$$

Binary Phase shift keying

- Where $0 \leq t \leq T_b$, and E_b is the transmitted signal energy per bit
- The carrier frequency is selected such that $f_c = \frac{n}{T_b}$ so that each bit contains an integral number of cycles
- From the pair of symbols $s_1(t)$ and $s_2(t)$ we can see only one basis function (carrier) is needed to represent both $s_1(t)$ and $s_2(t)$

Binary Phase shift keying

- The basis function is given by

$$\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t) \quad 0 \le t \le T_b$$

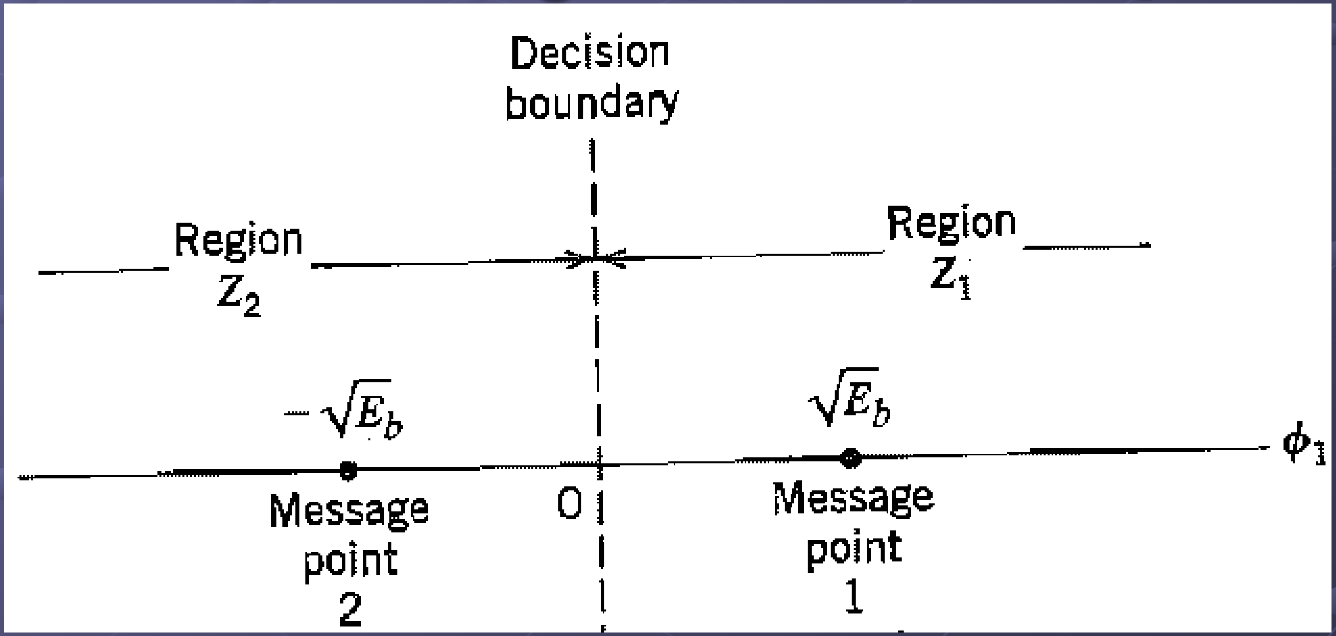
- Now we can rewrite
- $s_1(t) = \sqrt{E_b}\phi_1(t)$ and $s_2(t) = -\sqrt{E_b}\phi_1(t)$ on the interval $0 \le t \le T_b$

Signal constellation for binary Phase shift keying

- In order to draw the constellation diagram we need to find the projection of each transmitted symbol on the basis function
- The projection of the logic (1); $S_1(t)$; is given by $S_{11} = \int_0^{T_b} S_1(t)\phi_1(t)dt = +\sqrt{E_b}$
- The projection of the second symbol $S_2(t)$ on the basis function is given by $S_{21} = \int_0^{T_b} S_2(t)\phi_1(t)dt = -\sqrt{E_b}$

Signal constellation for binary Phase shift keying

- If we plot the transmitted symbols for BPSK we may got the following constellation diagram

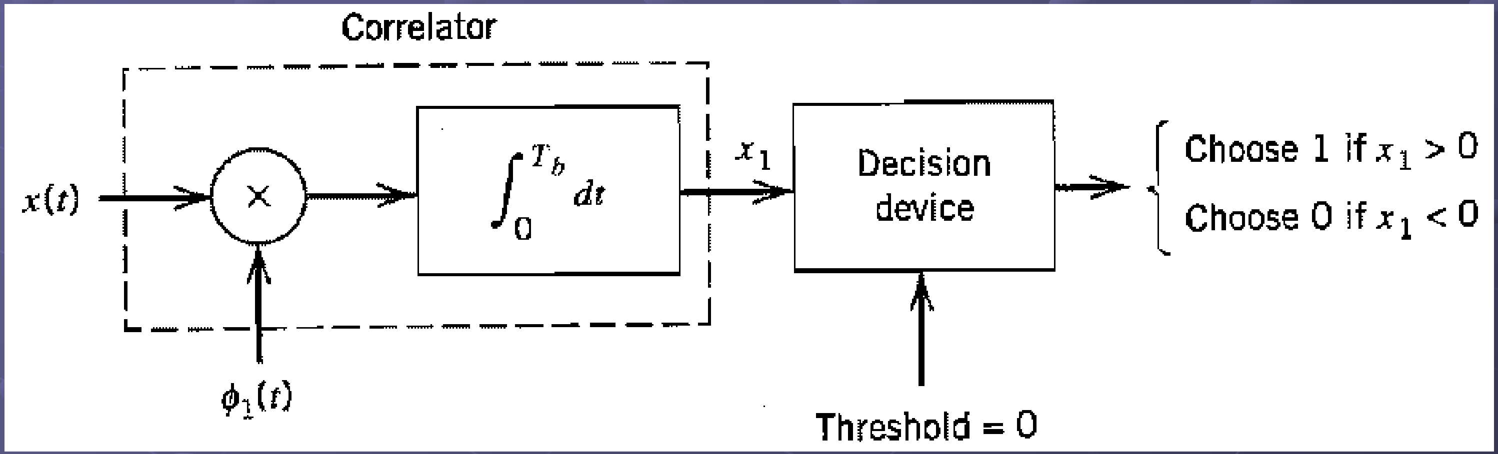


Error probability of BPSK

- In order to compute the error probability of BPSK we partition the constellation diagram of the BPSK (see slide 6) into two regions
- If the received symbol falls in region Z_1, the receiver decides in favor of symbol S_1 (logic 1) was received
- If the received symbol falls in region Z_2, the receiver decides in favor of symbol S_2 (logic 0) was received

Error probability of BPSK-Receiver model

- The receiver in the pass band can be modeled as shown



- The received signal vector $x(t) = s\,(t) + n(t)$

Error probability of BPSK

- The observable element x_1 (symbol zero was sent and the detected sample was read in zone 1) is given by

$$x_1 = \int_0^{T_b} x_1(t)\phi_1(t)dt$$

$$x_1 = \int_0^{T_b} (s_2(t) + n(t))\phi_1(t)dt$$

$$x_1 = \int_0^{T_b} s_2(t)\phi_1(t)dt = S_{21} = -\sqrt{E_b}$$

Error probability of BPSK

- To calculate the probability of error that symbol 0 was sent and the receiver detect 1 mistakenly in the presence of AWGN with $\sigma^2{}_x = \frac{N_0}{2}$, we need to find the conditional probability density of the random variable x_1, given that symbol 0, $s_2(t)$, was transmitted as shown below

$$f_{X_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 - s_{21})^2\right]$$

$$= \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 + \sqrt{E_b})^2\right]$$

Error probability of BPSK

- The conditional probability of the receiver deciding in favor of symbol 1, given that symbol zero was transmitted is given by

$$p_{10} = \int_0^\infty f_{X_1}(x_1|0)\,dx_1$$

$$= \frac{1}{\sqrt{\pi N_0}} \int_0^\infty \exp\left[-\frac{1}{N_0}(x_1 + \sqrt{E_b})^2\right] dx_1$$

Error probability of BPSK

- By letting

$$z = \frac{1}{\sqrt{N_0}}\left(x_1 + \sqrt{E_b}\right)$$

- the above integral for p_{10} can be rewritten as

$$p_{10} = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b/N_0}}^{\infty} \exp(-z^2)\, dz = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Error probability of error

- In similar manner we can find probability of error that symbol 1 was sent and the receiver detect 0 mistakenly
- The average probability as we did in the baseband can be computed as

$$P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

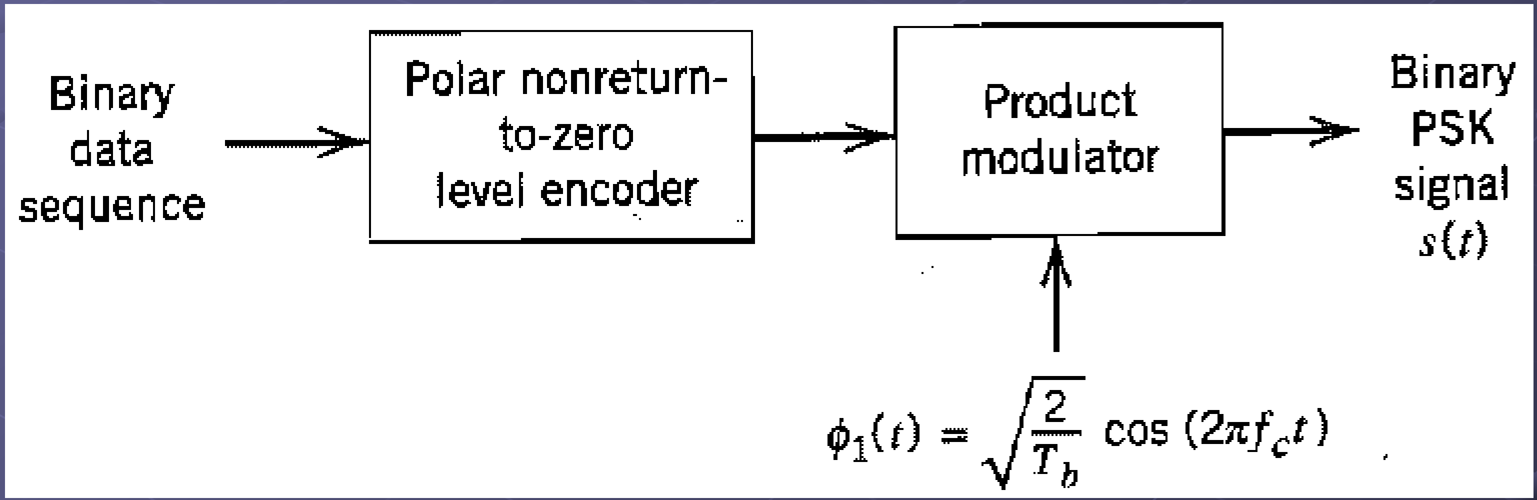
- This average probability is equivalent to the bit error rate

Generation of BPSK signals

- To generate a binary PSK signal we need to present the binary sequence in polar form
- The amplitude of logic 1 is $+\sqrt{E_b}$ whereas the amplitude of logic 0 is $-\sqrt{E_b}$
- This signal transmission encoding is performed by using polar NRZ encoder

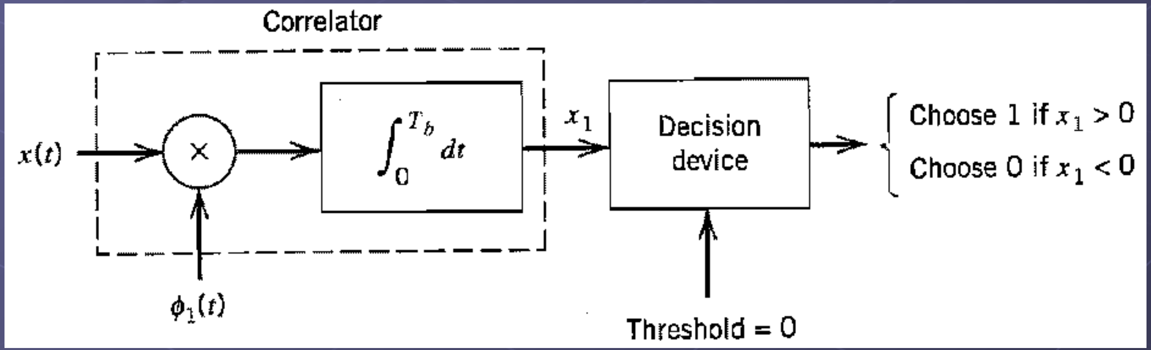
Generation of BPSK signals

- The resulting binary wave and the carrier (basis function) are applied to product modulator as shown below



Detection of BPSK signals

- To detect the original binary sequence we apply the received noisy PSK signal $x(t) = s\ (t) + n(t)$ to a correlator followed by a decision device as shown below



- The correlator works as a matched filter

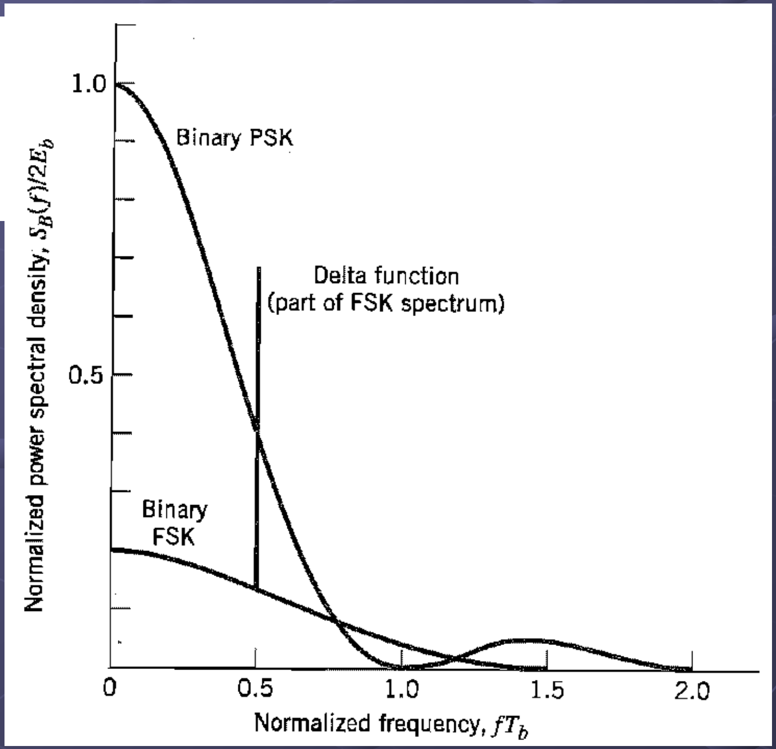
Power spectra of binary PSK signals

- The power spectral density of the binary PSK signal can be found as described for the bipolar NRZ signaling (see problem 3.11 (a) Haykin)
- This assumption is valid because the BPSK is generated by using bipolar NRZ signaling

Power spectra of binary PSK signals

- The power spectral density can be found as

$$S_B(f) = \frac{2E_b \sin^2(\pi T_b f)}{(\pi T_b f)^2}$$
$$= 2E_b \operatorname{sinc}^2(T_b f)$$



Bandwidth of the BPSK modulation scheme

- Since the BPSK modulation is similar to DSB_SC modulation, the bandwidth required for the transmission of BPSK signal is given by

$$BW = 2f_m$$

- Where f_m is the maximum frequency content of the BPSK signal
- If the bandwidth is defined as the null to null for the sinc function in slide 18, it follows that $f_m = R_b$, therefore the bandwidth for the BPSK is given by

$$BW = 2R_b = \frac{2}{T_b}$$

Quadrature phase shift keying QPSK

- In quadrature phase shift keying 4 symbols are sent as indicated by the equation

$$s_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} cos\left[2\pi f_c t + (2i-1)\frac{\pi}{4}\right] & 0 \le t \le T \\ 0 & elsewhere \end{cases}$$

- Where $i = 1, 2, 3, 4;$ E is the transmitted signal energy per symbol, and T is the symbol duration
- The carrier frequency is $\frac{n_c}{T}$ for some fixed integer n_c

Signal space diagram of QPSK

- If we expand the QPSK equation using the trigonometric identities we got the following equation

$$s_i(t) = \sqrt{E}\cos\left[(2i-1)\frac{\pi}{4}\right]\sqrt{\frac{2}{T}}\cos(2\pi f_c t) - \sqrt{E}\sin\left[(2i-1)\frac{\pi}{4}\right]\sqrt{\frac{2}{T}}\sin(2\pi f_c t)$$

$$= \sqrt{E}\cos\left[(2i-1)\frac{\pi}{4}\right]\phi_1(t) - \sqrt{E}\sin\left[(2i-1)\frac{\pi}{4}\right]\phi_2(t); \quad 0 \le t < T$$

- Which we can write in vector format as

$$\mathbf{s}_i = \begin{bmatrix} \sqrt{E}\cos(2i-1)\frac{\pi}{4} \\ -\sqrt{E}\sin(2i-1)\frac{\pi}{4} \end{bmatrix}$$

Signal space diagram of QPSK

- There are four message points defined by

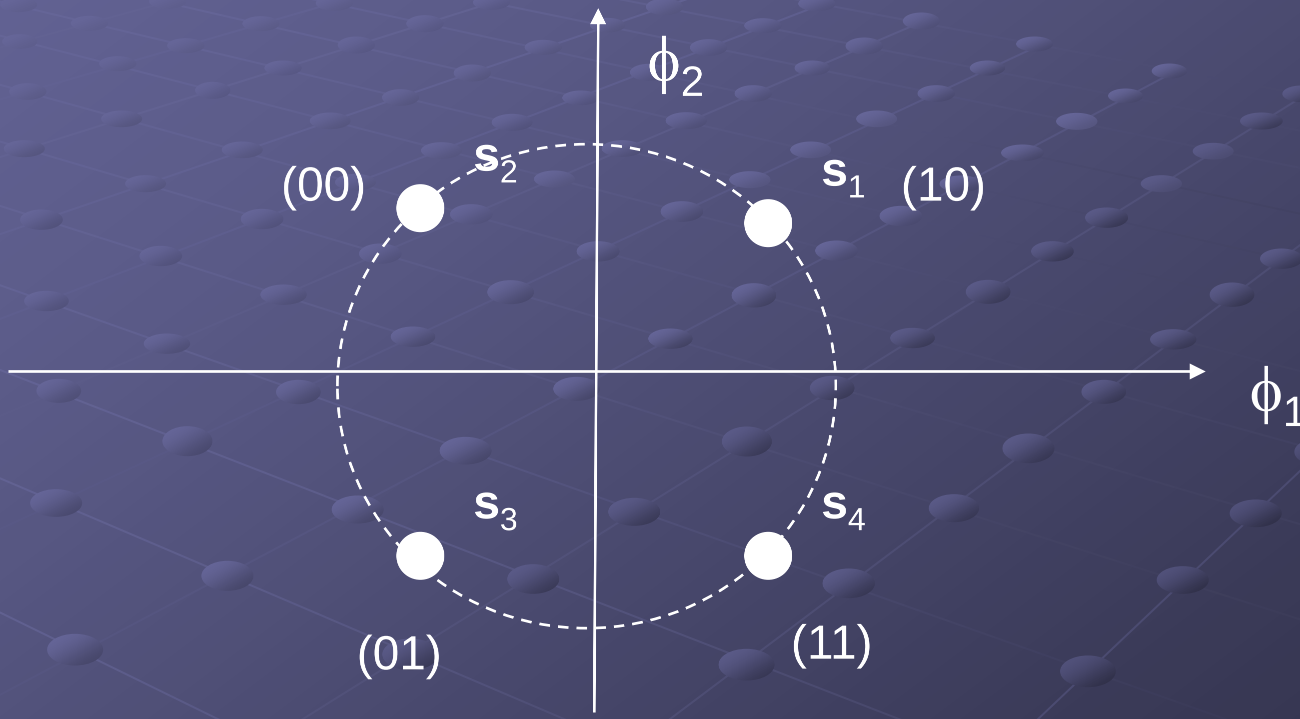
$$\mathbf{s}_i = \begin{bmatrix} \sqrt{E}\cos\left((2i-1)\dfrac{\pi}{4}\right) \\ -\sqrt{E}\sin\left((2i-1)\dfrac{\pi}{4}\right) \end{bmatrix}, \qquad i = 1, 2, 3, 4$$

- According to this equation , a QPSK has a two-dimensional signal constellation (i.e. $N = 2$ or two basis functions)

Detailed message points for QPSK

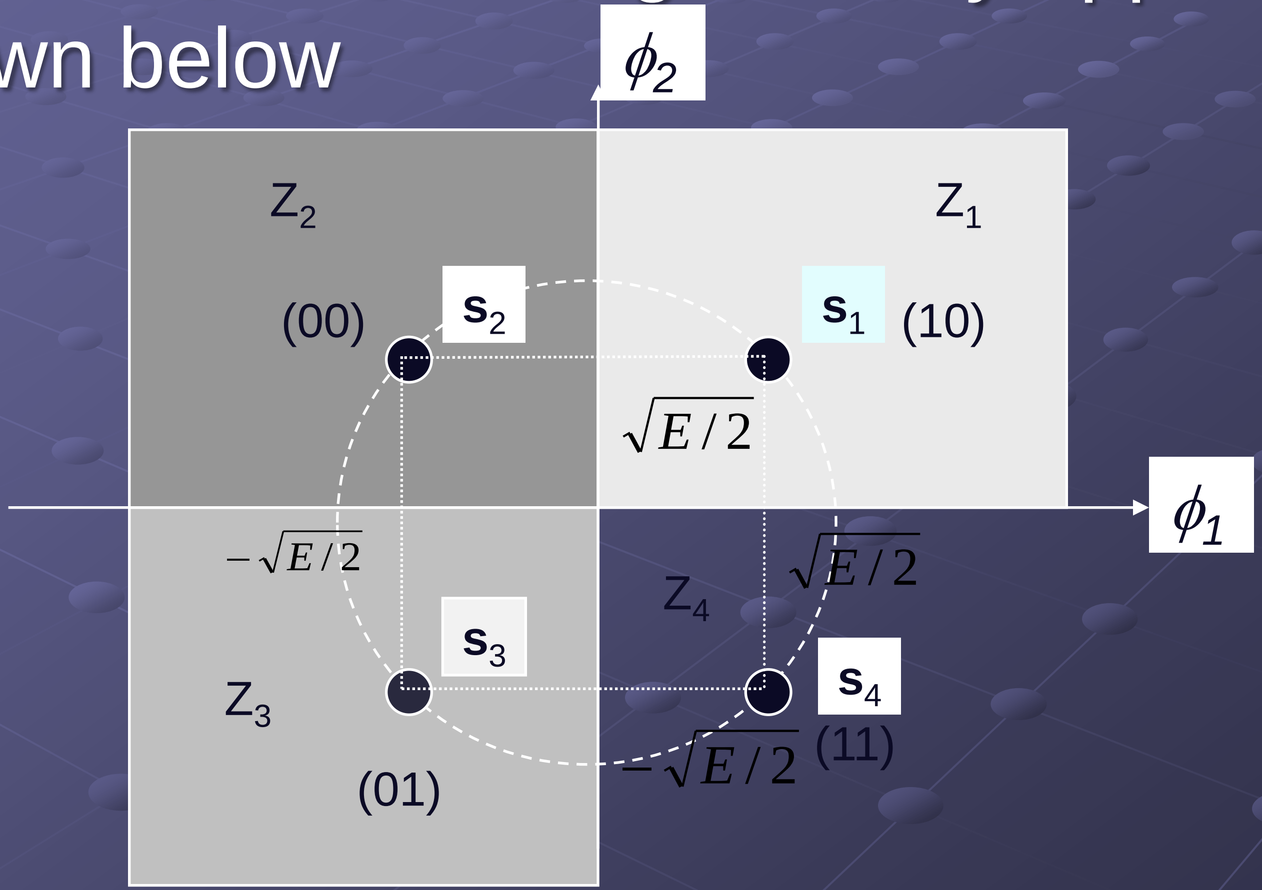
i	Input Dibit	Phase of QPSK signaling	Coordinate of Message point	
			s_{i1}	s_{i2}
1	10	$\pi/4$	$\sqrt{E/2}$	$-\sqrt{E/2}$
2	00	$3\pi/4$	$-\sqrt{E/2}$	$-\sqrt{E/2}$
3	01	$5\pi/4$	$-\sqrt{E/2}$	$\sqrt{E/2}$
4	11	$7\pi/4$	$\sqrt{E/2}$	$\sqrt{E/2}$

Signal space diagram of QPSK



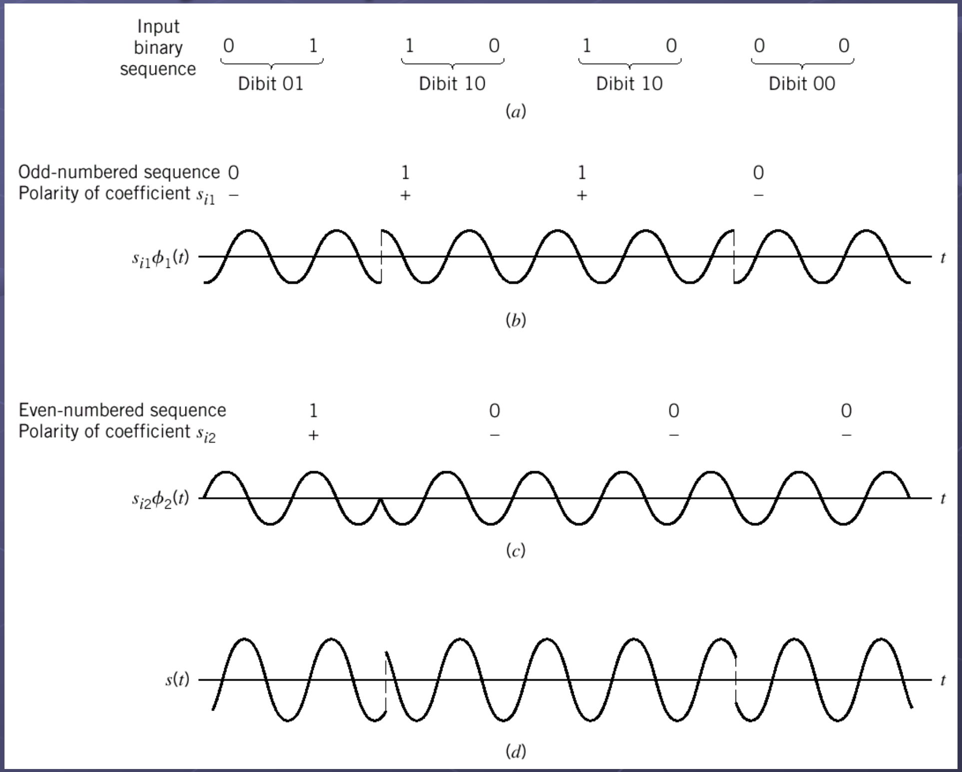
Signal space diagram of QPSK with decision zones

- The constellation diagram may appear as shown below



Example

- Sketch the QPSK waveform resulting from the input binary sequence 01101000
- solution



Error probability of QPSK

- In coherent QPSK, the received signal $x(t)$ is defined by

$$x(t) = s_i(t) + w(t) \quad \begin{Bmatrix} 0 \le t \le T \\ i = 1, 2, 3, 4 \end{Bmatrix}$$

- Where $w(t)$ is the sample function of AWGN with zero mean and power spectral density of $\sigma^2{}_x = \frac{N_0}{2}$

Error probability decision rule

- If the received signal point associated with the observation vector x falls inside region Z_1, the receiver decide that $s_1(t)$ was transmitted
- Similarly the receiver decides that $s_2(t)$ was transmitted if x falls in region Z_2
- The same rule is applied for $s_3(t)$ and $s_4(t)$

Error probability of QPSK

- We can treat QPSK as the combination of 2 independent BPSK over the interval $T = 2T_b$
- since the first bit is transmitted by ϕ_1 and the second bit is transmitted by ϕ_2
- Probability of error for each channel is given by $P' = \frac{1}{2}\operatorname{erfc}\left(\frac{d_{12}}{2\sqrt{N_0}}\right) = \frac{1}{2}\operatorname{erf}c\left(\sqrt{\frac{E}{2N_0}}\right)$

Error probability of QPSK

- If symbol is to be received correctly both bits must be received correctly
- Hence, the average probability of correct decision is given by $P_c = (1 - P')^2$
- Which gives the probability of errors equal to

$$P_e = 1 - P_C = \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) - \frac{1}{4}\operatorname{erfc}^2\left(\sqrt{\frac{E}{2N_0}}\right)$$

$$\approx \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)$$

Error probability of QPSK

- Since one symbol of QPSK consists of two bits, we have $E = 2Eb$ $P_e(\text{per symbol}) \approx \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$
- The above probability is the error probability per symbol
- With gray encoding the average probability of error per bit

$$P_e(\text{per bit}) = \frac{1}{2} P_e(\text{per symbol}) \approx \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

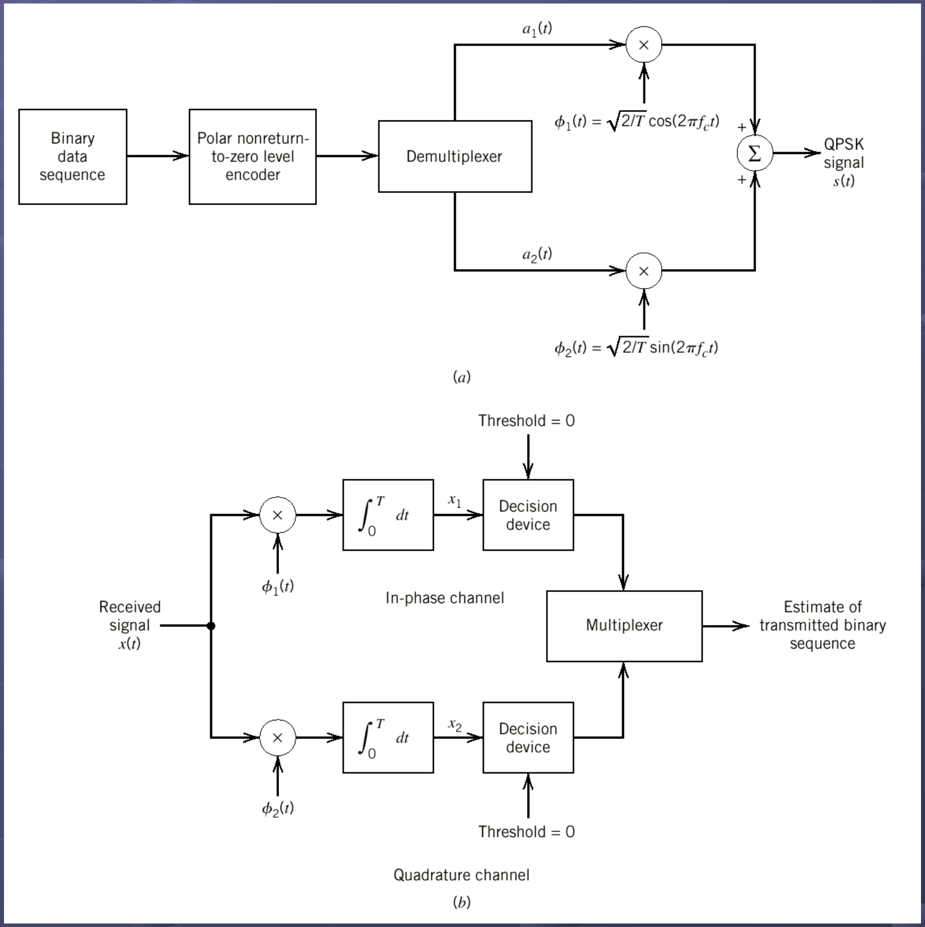
- Which is exactly the same as BPSK

Error probability of QPSK summery

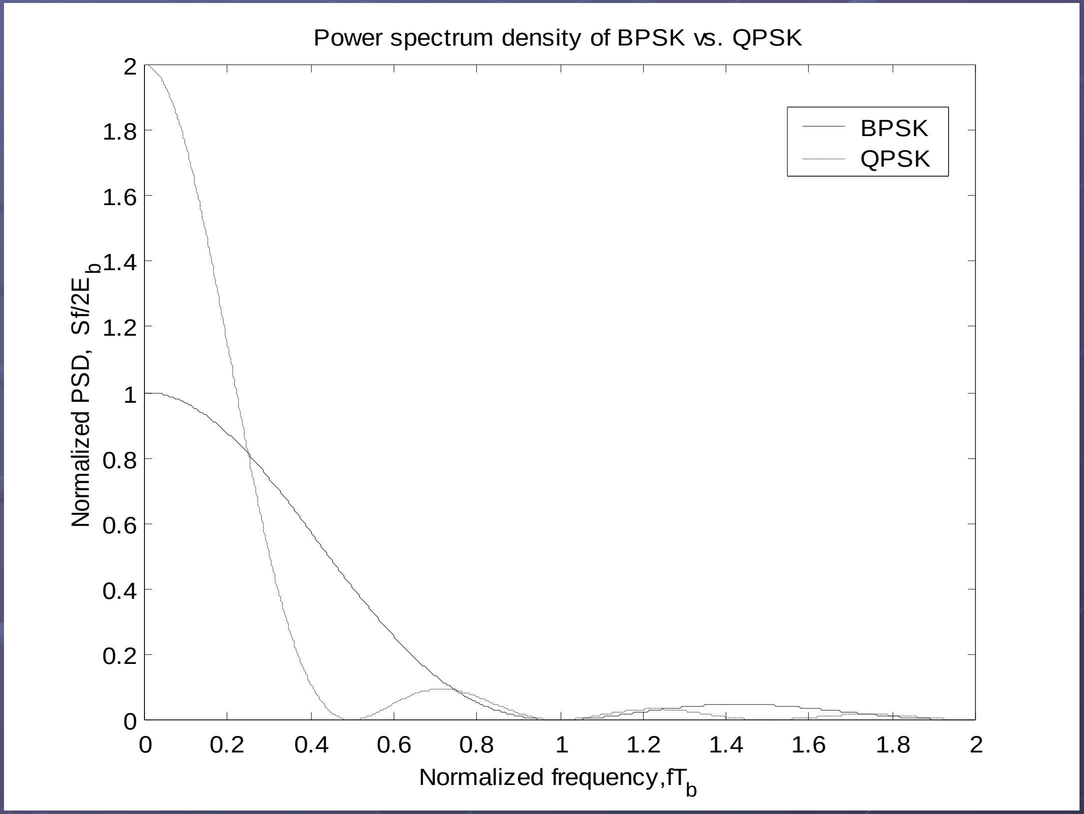
- We can state that a coherent QPSK system achieves the same average probability of bit error as a coherent PSK system for the same bit rate and the same $\frac{E_b}{N_0}$ but uses only half the channel bandwidth

Generation and detection of QPSK

- Block diagrams of (*a*) QPSK transmitter and (*b*) coherent QPSK receiver.



Power spectra of QPSK



Bandwidth of QPSK

- The bandwidth of a QPSK modulated signal is given by

$$BW = \frac{2}{T} = \frac{2}{\log_2 M \, T_b}$$

- Since M is 4, then the bandwidth QPSK signal is given by

$$BW = \frac{2}{\log_2 4 T_b} = R_b$$

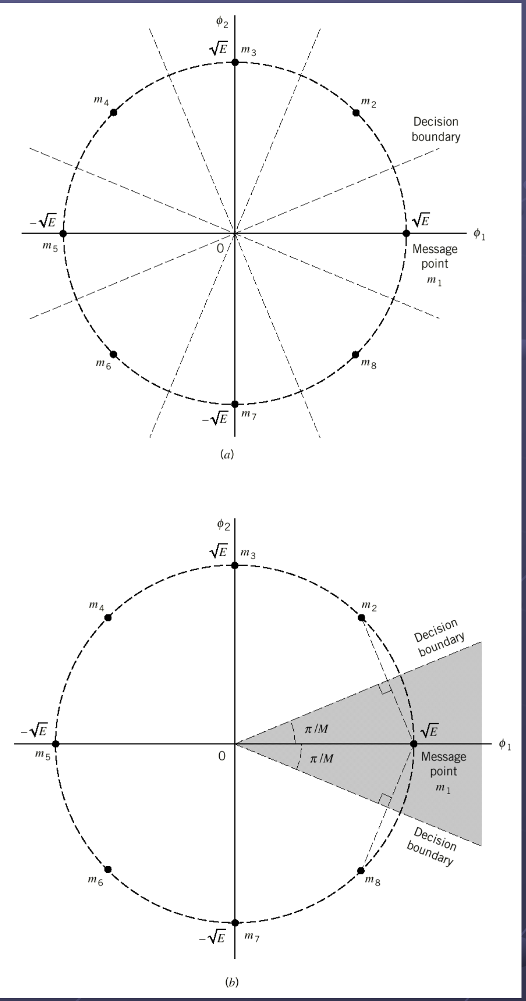
M-array PSK

- At a moment, there are M possible symbol values being sent for M different phase values, $\theta_i = 2(i-1)\pi / M$

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left(2\pi f_c t + \frac{2\pi}{M}(i-1)\right), \qquad i = 1,2,\ldots,M$$

M-array PSK

- Signal-space diagram for octa phase-shift keying (i.e., *M* = 8). The decision boundaries are shown as dashed lines.
- Signal-space diagram illustrating the application of the union bound for octa phase shift keying.



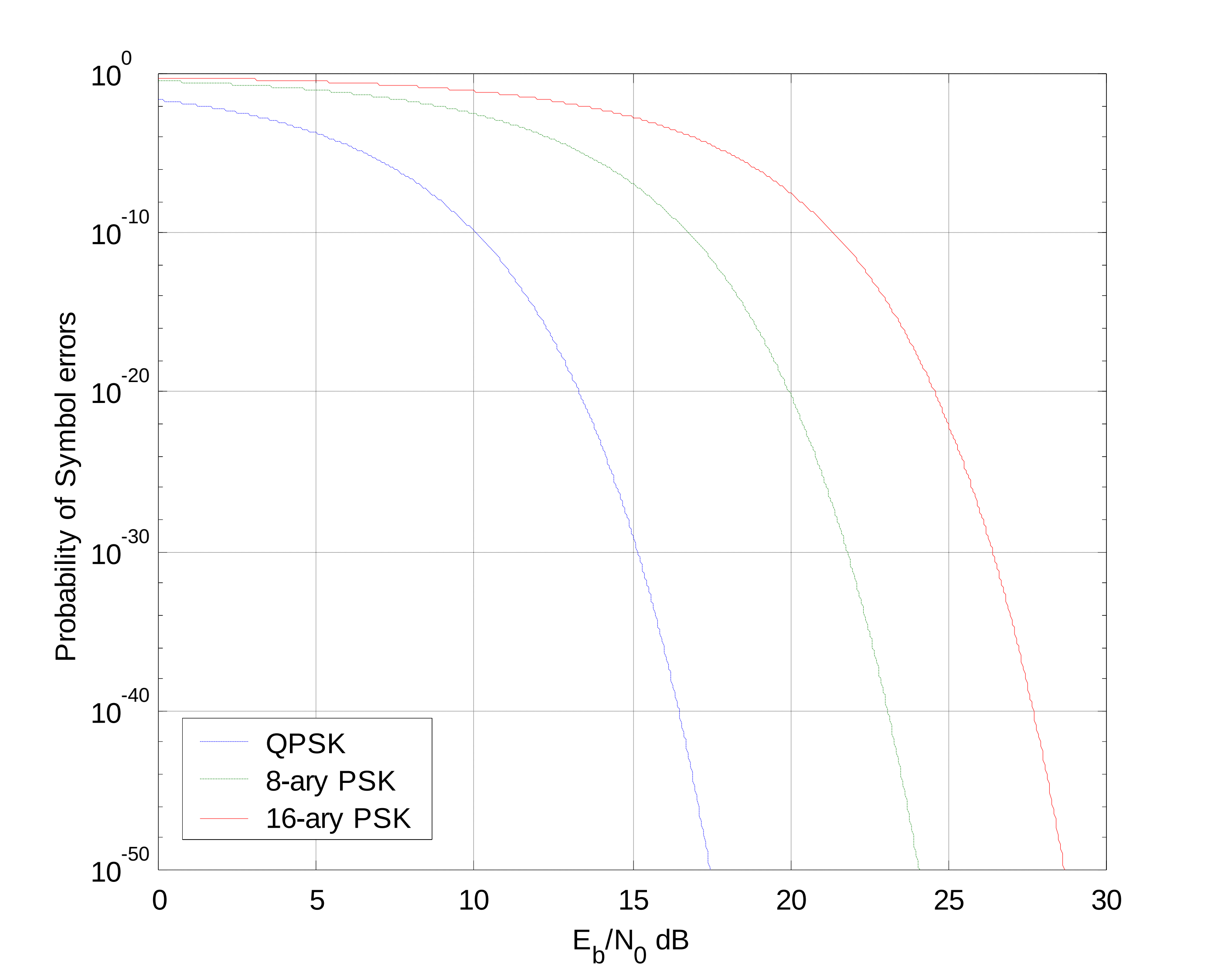
M-array PSK

- Probability of errors

$$\therefore d_{12} = d_{18} = 2\sqrt{E}\sin(\pi / M)$$

$$P_e \approx \text{erfc}\left(\sqrt{\frac{E}{N_0}}\sin(\pi / M)\right); \quad M \geq 4$$

M-ary PSK



Probability of Symbol errors
10^{0}
10^{-10}
10^{-20}
10^{-30}
10^{-40}
10^{-50}
0
5
10
15
20
25
30
E_b/N_0 dB
QPSK
8-ary PSK
16-ary PSK

M-array PSK

- Power Spectra (M-array)

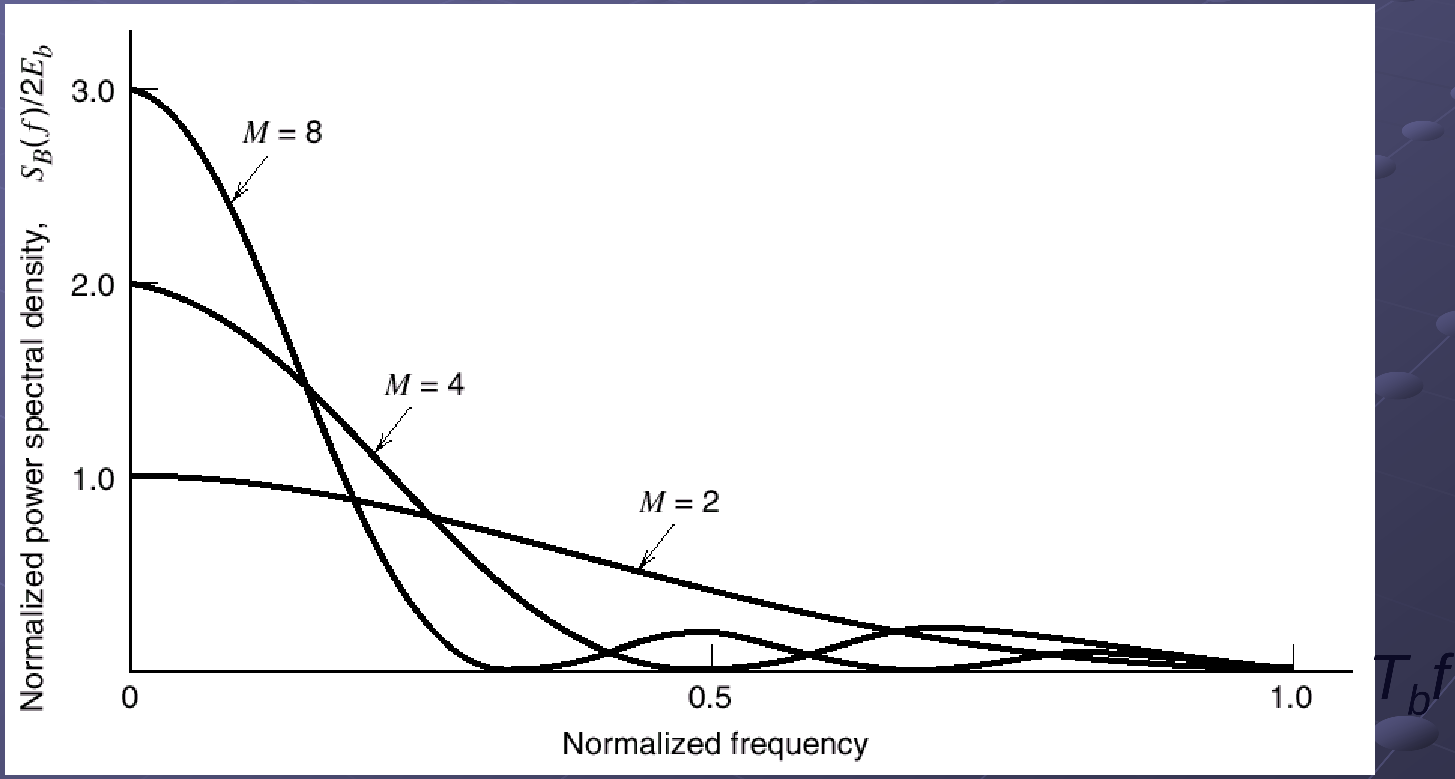
$$S_{PSK}(f) = 2E\,\mathrm{sinc}^2(Tf)$$
$$= 2E_b \log_2 M\,\mathrm{sinc}^2(T_b f \log_2 M)$$

- M=2, we have

$$S_{BPSK}(f) = 2E_b\,\mathrm{sinc}^2(T_b f)$$

M-array PSK

- Power spectra of *M*-ary PSK signals for M = 2, 4, 8.



M-array PSK

- Bandwidth efficiency:
  - We only consider the bandwidth of the main lobe (or null-to-null bandwidth)

$$B = \frac{2}{T} = \frac{2}{T_b \log_2 M} = \frac{2R_b}{\log_2 M}$$

  - Bandwidth efficiency of M-ary PSK is given by

$$\rho = \frac{R_b}{B} = \frac{R_b}{2R_b} \log_2 M = 0.5 \log_2 M$$

M-ary QAM

- QAM = Quadrature Amplitude Modulation
- Both Amplitude and phase of carrier change according to the transmitted symbol, m_i

$$s_i(t) = \sqrt{\frac{2E_0}{T}} a_i \cos(2\pi f_c t) - \sqrt{\frac{2E_0}{T}} b_i \sin(2\pi f_c t); \quad 0 < t \le T$$

where a_i and b_i are integers, E_0 is the energy of the signal with the lowest amplitude

M-ary QAM

- Again, we have

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos 2\pi f_c t \quad ;0 < t \le T$$

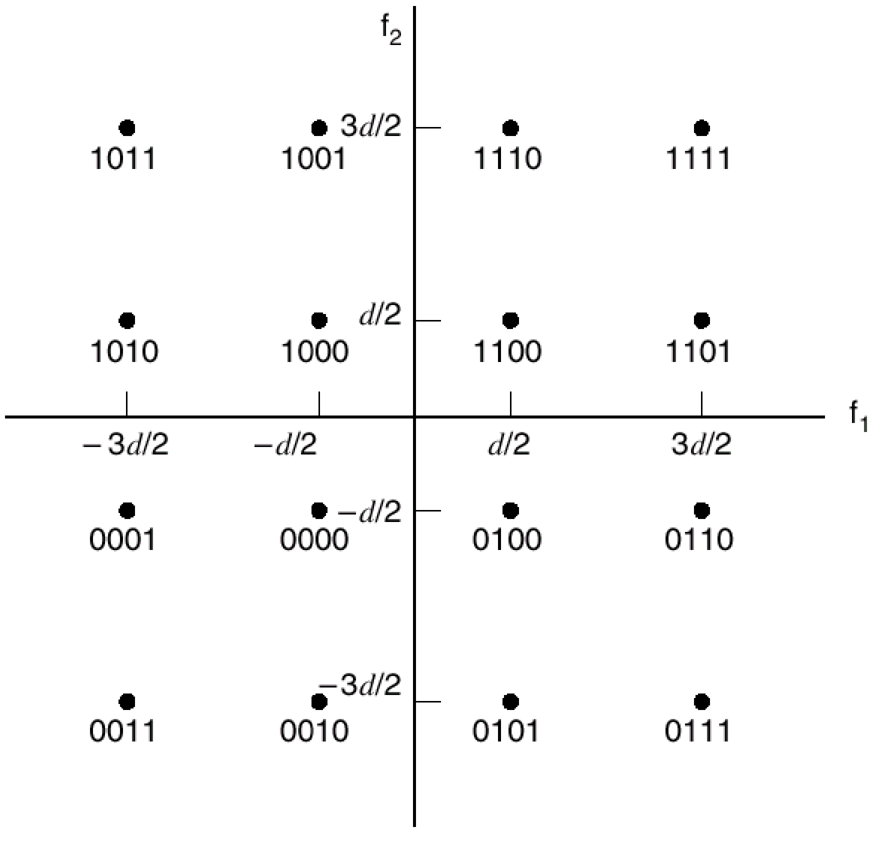
$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin 2\pi f_c t \quad ;0 < t \le T$$

as the basis functions

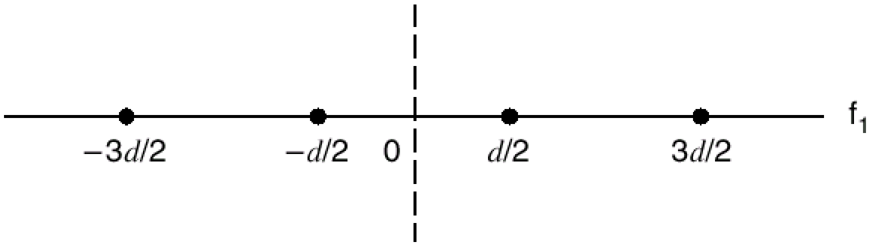
There are two QAM constellations, square constellation and rectangular

M-ary QAM

- QAM square Constellation
  - Having even number of bits per symbol, denoted by $2n$
  - $M = L \times L$ possible values
  - Denoting $L = \sqrt{M}$



(a)



(b)

L-ary, 4-PAM

16-QAM

16-QAM

- Calculation of Probability of errors
  - Since both basis functions are orthogonal, we can treat the 16-QAM as combination of two 4-ary PAM systems.
  - For each system, the probability of error is given by

$$P_e' = \left(1 - \frac{1}{L}\right) erfc\left(\frac{d}{2\sqrt{N_0}}\right) = \left(1 - \frac{1}{\sqrt{M}}\right) erfc\left(\sqrt{\frac{E_0}{N_0}}\right)$$

16-QAM

- A symbol will be received correctly if data transmitted on both 4-ary PAM systems are received correctly. Hence, we have

$$P_c(symbol) = (1 - P_e')^2$$

- Probability of symbol error is given by

$$P_e(symbol) = 1 - P_c(symbol) = 1 - (1 - P_e')^2$$
$$= 1 - 1 + 2P_e' - (P_e')^2 \approx 2P_e'$$

16-QAM

- Hence, we have

$$P_e(symbol) = 2\left(1 - \frac{1}{\sqrt{M}}\right) erfc\left(\sqrt{\frac{E_0}{N_0}}\right)$$

- But because average energy is given by

$$E_{av} = 2\left[\frac{2E_0}{L}\sum_{i=1}^{L/2}(2i-1)^2\right] = \frac{2(M-1)E_0}{3}$$

- We have

$$P_e(symbol) = 2\left(1 - \frac{1}{\sqrt{M}}\right) erfc\left(\sqrt{\frac{3E_{av}}{2(M-1)N_0}}\right)$$