Fluid Statics

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Fluid statics is used to determine the forces acting on floating or submerged bodies.

The design of many engineering systems such as water dams and liquid storage tanks requires the determination of the forces acting on the surfaces using fluid statics.

The complete description of the resultant hydrostatic force acting on a submerged surface requires the determination of the magnitude, the direction, and the line of action of the force.
Hydrostatic Forces on Submerged Plane Surfaces

- A plate exposed to a liquid is subjected to fluid pressure distributed over its surface.

- On a plane surface, the hydrostatic forces form a system of parallel forces, and we often need to determine the magnitude of the force and its point of application, which is called the center of pressure.

- In most cases, the other side of the plate is open to the atmosphere (such as the dry side of a gate), and thus atmospheric pressure acts on both sides of the plate, yielding a zero resultant.
Hydrostatic Forces on Submerged Plane Surfaces

- In such cases, it is convenient to \textbf{subtract} atmospheric pressure and work with the gage pressure only.

- For example

\[ P_{\text{gage}} = \rho gh \] at the bottom of the lake.
Hydrostatic Forces on Submerged Plane Surfaces

- Consider the top surface of a flat plate of arbitrary shape completely submerged in a liquid, as shown in the figure.
- The plane of this surface intersects the horizontal free surface with an angle $\theta$, and we take the line of intersection to be the x-axis.
- The absolute pressure above the liquid is $P_0$, which is the local atmospheric pressure $P_{\text{atm}}$ if the liquid is open to the atmosphere.
Hydrostatic Forces on Submerged Plane Surfaces

\[ P_{\text{avg}} = P_C = P_{\text{atm}} + \rho gh_C \]
Hydrostatic Forces on Submerged Plane Surfaces

- Then the absolute pressure at any point on the plate is:
  \[ P = P_0 + \rho gh = P_0 + \rho gy \sin \theta \]
  where \( h \) is the vertical distance of the point from the free surface and \( y \) is the distance of the point from the x-axis.

- The resultant hydrostatic force \( F_R \) acting on the surface is determined by integrating the force \( P \, dA \) acting on a differential area \( dA \) over the entire surface area:
  \[ F_R = \int_A P \, dA \]
Hydrostatic Forces on Submerged Plane Surfaces

The resultant hydrostatic force $F_R$

$F_R = \int_A P \, dA = \int_A (P_0 + \rho g y \sin \theta) \, dA = P_0 A + \rho g \sin \theta \int_A y \, dA$

But the first moment of area $\int_A y \, dA$ is related to the $y$-coordinate of the centroid (or center) of the surface by:

$y_c = \frac{1}{A} \int_A y \, dA$

Substituting:

$F_R = (P_0 + \rho g y_c \sin \theta) A = (P_0 + \rho g h_c) A = P_C A = P_{\text{avg}} A$

where $P_0 + \rho g h_c$ is the pressure at the centroid of the surface and $h_c = g y_c \sin \theta$ is the vertical distance of the centroid from the free surface of the liquid
Next we need to determine the line of action of the resultant force $F_R$.

The line of action of $F_R$, in general, *does not pass through the centroid of the surface* it lies underneath where the pressure is higher.

The point of intersection of the line of action of $F_R$ and the surface is the *center of pressure*.
Hydrostatic Forces on Submerged Plane Surfaces

- The vertical location of the line of action is determined by equating the moment of \( F_R \) to the moment of the distributed pressure force about the x-axis. It gives:

\[
y_pF_R = \int_A yP \, dA = \int_A y(P_0 + \rho g y \sin \theta) \, dA = P_0 \int_A y \, dA + \rho g \sin \theta \int_A y^2 \, dA
\]

or

\[
y_pF_R = P_0 y_C A + \rho g \sin \theta I_{xx.0}
\]

where \( y_p \) is the distance of the center of pressure from the x-axis (point 0 in the figure) and \( I_{xx.0} = \int_A y^2 \, dA \) is the second moment of area (the area moment of inertia)
Hydrostatic Forces on Submerged Plane Surfaces

- However, we need to use the second moments of area about the axes passing through the centroid. We use the parallel axes theorem:

\[ I_{xx,0} = I_{xx,C} + y_C^2 A \]

where \( I_{xx,C} \) is the \textit{second moment of area about the x-axes passing through the centroid}.

- Rearranging the equations gives:

\[ y_P = y_C + \frac{I_{xx,C}}{\left[ y_C + P_0 / \rho g \sin \theta \right] A} \]

- If \textit{no symmetry}, then \( x_P \) should be determined:

\[ x_P = x_C + \frac{I_{xy,C}}{\left[ y_C + P_0 / \rho g \sin \theta \right] A} \]
Hydrostatic Forces on Submerged Plane Surfaces

Resultant force: \( F = p_{CG} A \)

Free surface \( p = p_a \)

\( h(x, y) \)

\( h_{CG} \)

\( \theta \)

\( \xi = \frac{h}{\sin \theta} \)

Plan view of arbitrary plane surface

Side view

\( dA = dx \, dy \)

CG

CP
Hydrostatic Forces on Submerged Plane Surfaces

For $P_0 = 0$, which is usually the case when the atmospheric pressure is ignored, equations of $F_R$, $x_p$, and $y_p$ become:

\[ F_R = \rho gh_c A \]

\[ y_P = y_C + \frac{I_{xx,C}}{y_C A} \]

\[ x_P = x_C + \frac{I_{xy,C}}{y_C A} \]
I_{xx,C} values for some common areas

(a) Rectangle

\[ A = ba \]
\[ I_{xc} = \frac{1}{12} ba^3 \]
\[ I_{yc} = \frac{1}{12} ab^3 \]
\[ I_{xyc} = 0 \]

(b) Circle

\[ A = \pi R^2 \]
\[ I_{xc} = I_{yc} = \frac{\pi R^4}{4} \]
\[ I_{xyc} = 0 \]

(c) Semicircle

\[ A = \frac{\pi R^2}{2} \]
\[ I_{xc} = 0.1098R^4 \]
\[ I_{yc} = 0.3927R^4 \]
\[ I_{xyc} = 0 \]

(d) Triangle

\[ A = \frac{ab}{2} \]
\[ I_{xc} = \frac{ba^3}{36} \]
\[ I_{xyc} = \frac{ba^2}{72}(b - 2d) \]
Centroids and Centroidal Moments of Inertia

(a) \[ A = bL \]
\[ I_{xx} = \frac{bL^3}{12} \]
\[ I_{xy} = 0 \]

(b) \[ A = \pi R^2 \]
\[ I_{xx} = \frac{\pi R^4}{4} \]
\[ I_{xy} = 0 \]

(c) \[ A = \frac{bL}{2} \]
\[ I_{xx} = \frac{bL^3}{36} \]
\[ I_{xy} = \frac{b(b-2s)L^2}{72} \]

(d) \[ A = \frac{\pi R^2}{2} \]
\[ I_{xx} = 0.10976R^4 \]
\[ I_{xy} = 0 \]
Example

An *elliptical* gate covers the end of a pipe 4 m in diameter. If the gate is hinged at the top, what normal force \( F \) is required to open the gate when water is 8 m deep above the top of the pipe and the pipe is open to the atmosphere on the other side? Neglect the weight of the gate.

\[
\text{Area} = \pi \times a \times b
\]
Example

- Compute $F_R$
  
  $F_R = 1,000 \times 9.81 \times (8 + \frac{1}{2} \times 4) \times (\pi \times 2 \times 2.5) = 1.541 \text{ MN}$

- The centroid is located at $y_p = 12.5 \text{ m}$
  
  $I_{xx,C} = \pi a^3 b/4 = \pi \times 2.5^3 \times 2/4 = 24.54 \text{ m}^4$
  
  The center of pressure is located at $y_C = 12.5 + 24.54/(12.5 \times \pi \times 2 \times 2.5) = 12.625 \text{ m}$ which implies that the center of pressure is at a distance of 0.125 m down from the centroid

- To obtain $F$, take moments about the hinge:
  
  $1.541 \times 106 \times (2.5 + 0.125) - F \times 5 = 0$
  
  $F = 809 \text{ kN}$
Example

- Determine the resultant force \( F_R \) due to water acting on the 3 m by 6 m rectangular area AB shown in the figure.

- \( F_R = 1,000 \times 9.79 \times (4+\frac{1}{2}\times6) \times (6 \times 3) \times 0.001 = 1,234 \text{ kN} \)

- \( l_{xx,C} = 3 \times 6^3/12 = 5.26 \text{ m}^4 \)
- \( y_C = 7 + 5.26/[7 \times (3 \times 6)] = 7.43 \text{ m from } O_1 \)
Example

Determine the resultant force due to water acting on the 4 m by 6 m triangular area CD shown in the figure. The apex of the triangle is at C.

\[
F_{CD} = (9.79) \left[ 3 + \left( \frac{2}{3} \times \sin 45^\circ \times 6 \right) \right] \left( \frac{1}{2} \times 4 \times 6 \right) = 685 \text{ kN}
\]

This force acts at a distance \( y_{cp} \) from axis \( O_2 \) and is measured along the plane of the area \( CD \).

\[
y_{cp} = \frac{(4) (6^3) / 36}{(5.83 / \sin 45^\circ) \left( \frac{1}{2} \times 4 \times 6 \right)} + \frac{5.83}{\sin 45^\circ} = 8.49 \text{ m from axis } O_2
\]
Example

A vertical gate is 5 m wide and has water at a depth of 75 m on one side and to a depth of 3 m on the other side. Find the resultant horizontal force on the gate and the position of its line of action.
Example

$R_1$ is the resultant force on the left-hand side and $R_2$ is the resultant force on the right-hand side.

Area of left-hand water face = $A_1 = BH$

Depth to centroid of l.-h. face = $\bar{y}_1 = \frac{1}{2}H$

$$R_1 = \rho g A_1 \bar{y}_1 = \frac{1}{2} \rho g BH^2$$

and acts at $\frac{1}{3}H$ from the bottom.

Similarly $R_2 = \frac{1}{2} \rho g Bh^2$ and acts at $\frac{1}{3}h$ from the bottom. $R_1$ and $R_2$ have a resultant force $R$ acting at a height $x$ from the bottom. Taking moments about the bottom of the gate
Example

\[ Rx = R_1 \times \frac{1}{3}H - R_2 \times \frac{1}{3}h = \frac{1}{6} \rho g B H^3 - \frac{1}{6} \rho g B h^3 \]

But

\[ R = R_1 - R_2 = \frac{1}{2} \rho g B (H^2 - h^2) \]

\[ \therefore x = \frac{\frac{1}{6} \rho g B (H^3 - h^3)}{\frac{1}{2} \rho g B (H^2 - h^2)} = \frac{H^3 - h^3}{H^2 - h^2} \times \frac{1}{3} \]

\[ = \frac{H^2 + Hh + h^2}{3(H + h)} \]

Putting \( H = 7.5 \text{ m}, h = 3 \text{ m}, B = 5 \text{ m} \) and \( \rho = 10^3 \text{ kg/m}^3 \)

Resultant force \( = R = \frac{1}{2} \rho g B (H^2 - h^2) \]

\[ = \frac{1}{2} (10^3 \times 9.81 \times 5)(7.5^2 - 3^2) \]

\[ = 1160 \text{ kN} \]

Resultant acts at \( x \) from the bottom given by

\[ x = \frac{H^2 + Hh + h^2}{3(H + h)} = \frac{7.5^2 + 7.5 \times 3 + 3^2}{3(7.5 + 3)} \]

\[ = 2.79 \text{ m from bottom of gate} \]
Example

Water rises to level E in the pipe attached to tank ABCD in the figure. Neglecting the weight of the tank and riser pipe, (a) determine and locate the resultant force acting on area AB, which is 8 ft wide; (b) compute the total force on the bottom of the tank; and (c) compare the total weight of the water with the result in (b) and explain the difference.
Example

(a) The depth of the center of gravity of area $AB$ is 15 ft below the free surface of the water at $E$.

Then

$$F = \gamma h A = (62.4)(12 + 3)(6 \times 8) = 44,900 \text{ lb}$$

acting at distance

$$\gamma_{cp} = \frac{(8)(6^3)/12}{(15)(6 \times 8)} + 15 = 15.20 \text{ ft from } O$$

(b) The pressure on the bottom $BC$ is uniform; hence the force

$$F = pA = (\gamma h)A = (62.4)(18)(20 \times 8) = 179,700 \text{ lb}$$

(c) The total weight of the water is $W = (62.4)[(20 \times 6 \times 8) + (12 \times 1)] = 60,700 \text{ lb}$.

A free body of the lower part of the tank (cut by a horizontal plane just above level $BC$) will indicate a downward force on area $BC$ of 179,700 lb, vertical tension in the walls of the tank, and the reaction of the supporting plane. The reaction must equal the total weight of water or 60,700 lb. The tension in the walls of the tank is caused by the upward force on the top $AD$ of the tank, which is

$$F_{AD} = (\gamma h)A = (62.4)(12)(160 - 1) = 119,000 \text{ lb upward}$$

An apparent paradox is thus clarified since, for the free body considered, the sum of the vertical forces is zero, i.e.,

$$179,700 - 60,700 - 119,000 = 0$$

and hence the condition for equilibrium is satisfied.
Example

The 2-m-diameter gate AB in the figure swings about a horizontal pivot C located 40 mm below the center of gravity. To what depth \( h \) can the water rise without causing an unbalanced clockwise moment about pivot C?
Example

If the center of pressure and axis C should coincide, there would be no unbalanced moment acting or the gate. Evaluating the center of pressure distance,

$$y_{cp} = \frac{l_{cg}}{y_{cg} A} + y_{cg} = \frac{\pi d^4/64}{y_{cg}(\pi d^2/4)} + y_{cg}$$

Then

$$y_{cp} - y_{cg} = \frac{\pi 2^4/64}{(h + 1)(\pi 2^2/4)} = \frac{40}{1000} \text{ m (given)}$$

from which $h = 5.25$ m above $A$. 
Example

The cubic tank shown in the figure is half full of water. Find: (a) the pressure on the bottom of the tank, (b) the force exerted by the fluids on a tank wall, and (c) the location of the center of pressure on a wall.
Example

(a) \[ p_{\text{bott}} = p_{\text{air}} + \gamma_{\text{water}} h_{\text{water}} = 8 \text{ kN/m}^2 + (9.81 \text{ kN/m}^3)(1 \text{ m}) \]
\[ = 17.81 \text{ kN/m}^2 = 17.81 \text{ kPa} \quad \text{ANS} \]

(b) The force acting on the tank end is divided into two components, labeled A and B on the pressure distribution sketch. Component A has a uniform pressure distribution, due to the pressure of the confined air, which acts throughout the water:

\[ F_A = p_{\text{air}} A_{\text{air}} = (8 \text{ kN/m}^2)(4 \text{ m}^2) = 32.0 \text{ kN} \]

For component B, i.e., the varying water pressure distribution on the lower half of the tank wall, the centroid C of the area of application is at

\[ h_c = y_c = 0.5(1 \text{ m}) = 0.5 \text{ m below the water top surface}, \]

so, from Eq. (3.16),

\[ F_B = \gamma_{\text{water}} h_c A_{\text{water}} = 9.81(0.5)2 = 9.81 \text{ kN} \]

So the total force on the tank wall is

\[ F = F_A + F_B = 32.0 + 9.81 = 41.8 \text{ kN} \quad \text{ANS} \]
(c) The locations of the centers of pressure of the component forces, as distances $y_p$ below the water top surface, are

$$ (y_p)_A = 0 \text{ m} $$

below the water top surface, to the centroid of the 2-m-square area for the uniform air pressure.

$$ (y_p)_B = \frac{2}{3}h_{\text{water}} = \frac{2}{3}(1 \text{ m}) = 0.667 \text{ m} $$

below the water top surface for the varying pressure on the rectangular wetted wall area. We could also find this using Eq. (3.18) with $y_c = 0.5 \text{ m}$, $I_c = bh^3/12 = 2(1)^3/12 = 0.1667 \text{ m}^4$, and $A = bh = 2 \text{ m}^2$.

Taking moments:

$$ F(y_p) = F_A(y_p)_A + F_B(y_p)_B $$

from which $y_p = 0.1565 \text{ m}$ below the water top surface  

ANS
Gate AB in the figure is 4 ft wide and is hinged at A. Gage G reads – 2.17 psi, and oil of specific gravity 0.750 is in the right-hand tank. What horizontal force must be applied at B for equilibrium of gate AB?
Example

The forces acting on the gate due to the liquids must be evaluated and located. For the right-hand side,

\[ F_{oil} = \gamma h_{cg} A = (0.750 \times 62.4)(3)(6 \times 4) = 3370 \text{ lb to the left} \]

acting

\[ \gamma_{cp} = \frac{(4)(6^3)/12}{(3)(4 \times 6)} + 3 = 4.00 \text{ ft from } A \]

It should be noted that the pressure intensity acting on the right-hand side of rectangle \( AB \) varies linearly from zero gage to a value due to 6 ft of oil (\( p = \gamma h \) is a linear equation). Loading diagram \( ABC \) indicates this fact. For a rectangular area only, the center of gravity of this loading diagram coincides with the center of pressure. The center of gravity is located \( \left( \frac{2}{3} \right)(6) = 4 \text{ ft from } A \), as above.

For the left-hand side, it is necessary to convert the negative pressure due to the air to its equivalent in feet of the liquid, water.

\[ h = -\frac{p}{\gamma} = -\frac{2.17 \times 144 \text{ lb/ft}^2}{62.4 \text{ lb/ft}^3} = -5.01 \text{ ft} \]

This negative pressure head is equivalent to having 5.01 ft less of water above level \( A \). It is convenient and useful to employ an imaginary water surface (IWS) 5.01 ft below the real surface and solve the problem by direct use of basic equations. Thus,

\[ F_{water} = (62.4)(6.99 + 3)(6 \times 4) = 15,000 \text{ lb acting to the right at the center of pressure} \]
Example

For the submerged rectangular area, $y_{cp} = \frac{(4) \left(6^3\right)/12}{(9.99)(6 \times 4)} + 9.99 = 10.29$ ft from $O$, or the center of pressure is $(10.29 - 6.99) = 3.30$ ft from $A$.

In Fig. 3-5(b), the free-body diagram of gate $AB$ shows the forces acting. The sum of the moments about $A$ must equal zero. Taking clockwise as plus,

$$+3370 \times 4 + 6F - 15,000 \times 3.30 = 0$$
and $$F = 6000$$ lb to the left
Example

The tank in the figure contains oil and water. Find the resultant force on side A BC, which is 4 ft wide.
Example

The total force on \(ABC\) is equal to \((F_{AB} + F_{BC})\). Find each force, locate it, and, using the principle of moments, determine the position of the total force on side \(ABC\).

\[(a) \quad F_{AB} = (0.800 \times 62.4)(5)(10 \times 4) = 9980 \text{ lb acting at a point } \left(\frac{2}{3}\right)(10) \text{ ft from } A \text{ or } 6.67 \text{ ft down.}\]

The same distance can be obtained by formula as follows:

\[
y_{cp} = \frac{(4)(10^3)/12}{(5)(4 \times 10)} + 5 = 6.67 \text{ ft from } A
\]

\[(b) \quad \text{Water is acting on area } BC, \text{ and any superimposed liquid can be converted into an equivalent depth of water. Employ an imaginary water surface (IWS) for this second calculation, locating the IWS by changing } 10 \text{ ft of oil to } 0.800 \times 10 = 8 \text{ ft of water. Then}\]

\[
F_{BC} = (62.4)(8 + 3)(6 \times 4) = 16,470 \text{ lb acting at the center of pressure}
\]

\[
y_{cp} = \frac{(4)(6^3)/12}{(11)(4 \times 6)} + 11 = 11.27 \text{ ft from } O \quad \text{or} \quad (2 + 11.27) = 13.27 \text{ ft from } A
\]

The total resultant force = \(9980 + 16,470 = 26,450 \text{ lb acting at the center of pressure for the entire area. The moment of this total force = the sum of the moments of its two parts. Using } A \text{ as a convenient axis,}\)

\[26,450 \quad Y_{cp} = (9980)(6.67) + (16,470)(13.27) \quad \text{and} \quad Y_{cp} = 10.78 \text{ ft from } A\]
Example

- A heavy car plunges into a lake during an accident and lands at the bottom of the lake on its wheels.
- The door is 1.2 m high and 1 m wide, and the top edge of the door is 8 m below the free surface of the water.
- Determine the hydrostatic force on the door and the location of the pressure center, and discuss if the driver can open the door.
Example

- Few assumptions should be made in order to facilitate the solution:
  - The bottom surface of the lake is horizontal
  - The passenger cabin is well-sealed so that no water leaks inside
  - The door can be approximated as a vertical rectangular plate
  - The pressure in the passenger cabin remains at atmospheric value since there is no water leaking in and no compression of the air inside
The average pressure on the door is the pressure value at the centroid (midpoint) of the door and is determined to be:

\[ P_{\text{avg}} = \rho gh_c = 1,000 \times 9.81 \times (8 + 1.2/2) \times (1/1,000) = 84.4 \text{ kN/m}^2 \]

- \( F_R \) on the door = 84.4 \times (1 \times 1.2) = 101.3 kN
- To find the pressure center, we first compute \( I_{xx,C} \)

\[ I_{xx,C} = 1 \times 1.2^3/12 = 0.04 \text{ m}^4 \]

\[ y_p = y_c + I_{xx,C}/(y_c A) = 8.6 + 0.04/(8.6 \times 1 \times 1.2) = 8.603 \text{ m} \]
Example

- **Discussion.** A strong person can lift 100 kg, whose weight is 981 N or about 1 kN. Also, the person can apply the force at a point farthest from the hinges (1 m farther) for maximum effect and generate a moment of 1 kN . m

- The resultant hydrostatic force acts under the midpoint of the door, and thus a distance of 0.5 m from the hinges

- This generates a moment of 50.6 kN . m, which is about 50 times the moment the driver can possibly generate. Therefore, it is impossible for the driver to open the door of the car
For a submerged curved surface, the determination of the resultant hydrostatic force typically requires the integration of the pressure forces that change direction along the curved surface.

The easiest way to determine the resultant hydrostatic force \( F_R \) acting on a two-dimensional curved surface is to determine the horizontal and vertical components \( F_H \) and \( F_V \) separately.
Hydrostatic Forces on Submerged Curved Surfaces

- This is done by considering the free-body diagram of the liquid block enclosed by the curved surface and the two plane surfaces (one horizontal and one vertical) passing through the two ends of the curved surface.

- Note that the vertical surface of the liquid block considered is simply the projection of the curved surface on a vertical plane, and the horizontal surface is the projection of the curved surface on a horizontal plane.
Hydrostatic Forces on Submerged Curved Surfaces

- The resultant force acting on the curved solid surface is then equal and opposite to the force acting on the curved liquid surface (Newton’s third law).

- The weight of the enclosed liquid block of volume \( V \) is simply \( W = \rho g V \), and it acts downward through the centroid of this volume.

- Noting that the fluid block is in static equilibrium, the force balances in the horizontal and vertical directions give:

  \[
  \text{Horizontal force component on curved surface: } F_H = F_x
  \]
  \[
  \text{Vertical force component on curved surface: } F_V = F_y + W
  \]
Hydrostatic Forces on Submerged Curved Surfaces

Thus, we conclude that:

- The *horizontal* component of the hydrostatic force acting on a curved surface is equal (in both magnitude and the line of action) to the hydrostatic force acting on the vertical projection of the curved surface.

- The vertical component of the hydrostatic force acting on a curved surface is equal to the hydrostatic force acting on the horizontal projection of the curved surface, plus (minus, if acting in the opposite direction) the weight of the fluid block.
Hydrostatic Forces on Submerged Curved Surfaces

- The magnitude of the resultant hydrostatic force acting on the curved surface is:

\[ F_R = \sqrt{F_H^2 + F_V^2} \]

- And the tangent of the angle it makes with the horizontal is:

\[ \alpha = \frac{F_V}{F_H} \]
Example

- A long solid cylinder of radius 0.8 m hinged at point A is used as an automatic gate, as shown in the figure.

- When the water level reaches 5 m, the gate opens by turning about the hinge at point A.

- Determine (a) the hydrostatic force acting on the cylinder and its line of action when the gate opens and (b) the weight of the cylinder per m length of the cylinder.
Example

Horizontal force on vertical surface:

\[ F_H = F_x = P_{avg}A = \rho gh_c A = \rho g(s + R/2)A \]

\[ = (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(4.2 + 0.8/2 \text{ m})(0.8 \text{ m} \times 1 \text{ m}) \left( \frac{1 \text{ kN}}{1000 \text{ kg} \cdot \text{m/s}^2} \right) \]

\[ = 36.1 \text{ kN} \]

Vertical force on horizontal surface (upward):

\[ F_y = P_{avg}A = \rho gh_c A = \rho gh_{\text{bottom}} A \]

\[ = (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(5 \text{ m})(0.8 \text{ m} \times 1 \text{ m}) \left( \frac{1 \text{ kN}}{1000 \text{ kg} \cdot \text{m/s}^2} \right) \]

\[ = 39.2 \text{ kN} \]
Example

Weight of fluid block per m length (downward):

\[ W = mg = \rho g V = \rho g (R^2 - \pi R^2/4) (1 \text{ m}) \]

\[ = (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(0.8 \text{ m})^2(1 - \pi/4)(1 \text{ m}) \left( \frac{1 \text{ kN}}{1000 \text{ kg} \cdot \text{ m/s}^2} \right) \]

\[ = 1.3 \text{ kN} \]

Therefore, the net upward vertical force is

\[ F_v = F_y - W = 39.2 - 1.3 = 37.9 \text{ kN} \]

Then the magnitude and direction of the hydrostatic force acting on the cylindrical surface become

\[ F_R = \sqrt{F_H^2 + F_V^2} = \sqrt{36.1^2 + 37.9^2} = 52.3 \text{ kN} \]

\[ \tan \theta = F_V/F_H = 37.9/36.1 = 1.05 \rightarrow \theta = 46.4^\circ \]
(b) When the water level is 5 m high, the gate is about to open and thus the reaction force at the bottom of the cylinder is zero. Then the forces other than those at the hinge acting on the cylinder are its weight, acting through the center, and the hydrostatic force exerted by water. Taking a moment about point \( A \) at the location of the hinge and equating it to zero gives

\[
F_R R \sin \theta - W_{cyl} R = 0 \quad \rightarrow \quad W_{cyl} = F_R \sin \theta = (52.3 \text{ kN}) \sin 46.4^\circ = 37.9 \text{ kN}
\]
Hydrostatic Forces on Submerged Curved Surfaces

\[ F_V = W_1 + W_2 + W_{\text{air}} \]
Example

A dam has a parabolic shape $z/z_0 = (x/x_0)^2$ as shown in Fig. E2.8a, with $x_0 = 10$ ft and $z_0 = 24$ ft. The fluid is water, $\gamma = 62.4$ lbf/ft$^3$, and atmospheric pressure may be omitted. Compute the forces $F_H$ and $F_V$ on the dam and their line of action. The width of the dam is 50 ft.
Example

- **Solution steps for the horizontal component:** The vertical projection of the parabola lies along the z axis in Fig. E2.8b and is a rectangle 24 ft high and 50 ft wide. Its centroid is halfway down, or \( h_{CG} = 24/2 = 12 \) ft. Its area is \( A_{proj} = (24 \text{ ft})(50 \text{ ft}) = 1200 \text{ ft}^2 \). Then, from Eq. (2.26),

\[
F_H = \gamma h_{CG} A_{proj} = \left(62.4 \frac{\text{lbf}}{\text{ft}^3}\right)(12 \text{ ft})(1200 \text{ ft}^2) = 898,560 \text{ lbf} \approx 899 \times 10^3 \text{ lbf}
\]

The line of action of \( F_H \) is below the centroid of \( A_{proj} \), as given by Eq. (2.29):

\[
y_{CP, proj} = -\frac{I_{xx} \sin \theta}{h_{CG} A_{proj}} = -\frac{(1/12)(50 \text{ ft})(24 \text{ ft})^3 \sin 90^\circ}{(12 \text{ ft})(1200 \text{ ft}^2)} = -4 \text{ ft}
\]

Thus \( F_H \) is \( 12 + 4 = 16 \) ft, or two-thirds of the way down from the surface (8 ft up from the bottom).

- **Comments:** Note that you calculate \( F_H \) and its line of action from the *vertical projection* of the parabola, not from the parabola itself. Since this projection is *vertical*, its angle \( \theta = 90^\circ \).
Example

- **Solution steps for the vertical component:** The vertical force $F_V$ equals the weight of water above the parabola.

  The area and centroid are shown in Fig. E2.8b. The weight of this parabolic amount of water is

  \[
  F_V = \gamma A_{\text{section}} b = \left( 62.4 \frac{\text{lbf}}{\text{ft}^3} \right) \left[ \frac{2}{3} (24 \text{ ft})(10 \text{ ft}) \right] (50 \text{ ft}) = 499,200 \text{ lbf} \approx 499 \times 10^3 \text{ lbf}
  \]
Example

This force acts downward, through the centroid of the parabolic section, or at a distance $3x_0/8 = 3.75$ ft over from the origin, as shown in Figs. E2.8b,c. The resultant hydrostatic force on the dam is

$$F = (F_H^2 + F_V^2)^{1/2} = [(899 \times 10^3 \text{ lbf})^2 + (499 \times 10^3 \text{ lbf})^2]^{1/2} = 1028 \times 10^3 \text{ lbf} \text{ at } 29^\circ \text{ Ans.}$$

This resultant is shown in Fig. E2.8c and passes through a point 8 ft up and 3.75 ft over from the origin. It strikes the dam at a point 5.43 ft over and 7.07 ft up, as shown.
Example

Find an algebraic formula for the net vertical force $F$ on the submerged semicircular projecting structure CDE in the figure. The structure has uniform width $h$ into the paper. The liquid has specific weight $\gamma$. 
Example

- The net force is the difference between the upward force $F_L$ on the lower surface DE and the downward force $F_U$ on the upper surface CD.

- The force $F_U$ equals $\gamma$ times the volume ABDC above surface CD.

- The force $F_L$ equals $\gamma$ times the volume ABDEC above surface DE. The latter is clearly larger.

- The difference is $\gamma$ times the volume of the structure itself. Thus the net upward fluid force on the semicylinder is:

$$F = \gamma_{\text{fluid}} \times (\text{volume CDE}) = \gamma_{\text{fluid}} \times \frac{1}{2} \times \pi \times R^2 \times b$$
Buoyancy and Stability

- It is a common experience that an object feels lighter and weighs less in a liquid than it does in air.

- Also, objects made of wood or other light materials float on water.

- These and other observations suggest that a fluid exerts an upward force on a body immersed in it. This force that tends to lift the body is called the buoyant force and is denoted by $F_B$. 

![Gravity](image)

![Buoyant Force](image)
Buoyancy

- The buoyant force is caused by the increase of pressure in a fluid with depth.

- Consider, for example, a flat plate of thickness $h$ submerged in a liquid of density $\rho_f$ as shown in the figure.

- The area of the top (and also bottom) surface of the plate is $A$, and its distance to the free surface is $s$. 

\[ \text{Buoyant force} = \rho_f g (s + h) A \]
Buoyancy

- The pressures at the top and bottom surfaces of the plate are $\rho_f \times g \times s$ and $\rho_f \times g \times (h+s)$, respectively.

- Then:
  
  $F_{\text{top}} = \rho_f g s A$ acts downward on the top surface
  
  $F_{\text{bottom}} = \rho_f g (h+s) A$, acts upward on the bottom surface

- The difference between these two forces is a net upward force ($F_B$), which is the buoyant force:

  
  $F_B = F_{\text{bottom}} - F_{\text{top}} = \rho_f g (h+s) A - \rho_f g s A = \rho_f g h A = \rho_f g V$

  where $V = h A$ is the volume of the plate.
Buoyancy

- Thus, we conclude that the buoyant force acting on a body **immersed** in a fluid is equal to the **weight of the fluid displaced by the body**, (weight of the liquid that would be needed to occupy the volume of the body)

- The buoyant force acts upward through the **centroid of the displaced volume**
Buoyancy

\[ F_B = \gamma_{\text{fluid}} V_{\text{DHCK}} \]

\[ F_B = \gamma_{\text{fluid}} V_{\text{AKB}} \]
Buoyancy

A body immersed in a fluid:

(1) Remains at rest at any point in the fluid when its density is equal to the density of the fluid
(2) Sinks to the bottom when its density is greater than the density of the fluid
(3) Rises to the surface of the fluid and floats when the density of the body is less than the density of the fluid
Example

A block of concrete weighs 100 lbf in air and weighs only 60 lbf when immersed in freshwater (62.4 lbf/ft³). What is the average specific weight of the block?

- \( F_B \) equals the difference between the weights in air and in freshwater = 100 – 60 = 40 lbf

- In addition, \( F_B = \gamma \times \text{volume} = 62.4 \times V = 40 \)
  \( V = 0.641 \text{ ft}^3 \)

- The specific weight of the block = 100/0.641 = 156 lbf/ft³
Example

A crane is used to lower weights into the sea (density 1025 kg/m³) for an underwater construction project. Determine the tension in the rope of the crane due to a rectangular 0.4-m × 0.4-m × 3-m concrete block (density = 2,300 kg/m³) when it is (a) suspended in the air and (b) completely immersed in water.
Example

- Consider the free-body diagram of the concrete block. The forces acting on the concrete block in air are its weight and the upward pull action (tension) by the rope. These two forces must balance each other, and thus the tension in the rope must be equal to the weight of the block:

- $V = 0.4 \times 0.4 \times 3 = 0.48 \text{ m}^3$

  $F_{T,\text{air}} = W = \rho_{\text{concrete}} \ g \ V = 
  2,300 \times 9.81 \times 0.48 / 1,000 = 10.8 \text{ kN}$
Example

When the block is immersed in water, there is the additional force of buoyancy acting upward. The force balance in this case gives:

\[ F_B = \rho_f g V = \]
\[ 1,025 \times 9.81 \times 0.48 / 1,000 = 4.8 \text{ kN} \]

\[ F_{T,\text{water}} = W - F_B = 10.8 - 4.8 = 6 \text{ kN} \]
Example

- A spherical buoy has a diameter of 1.5 m, weighs 8.50 kN, and is anchored to the sea floor with a cable.

- Although the buoy normally floats on the surface, at certain times the water depth increases so that the buoy is completely immersed as illustrated.

- For this condition what is the tension of the cable?
Example

- We first draw a free-body diagram of the buoy where $F_B$ is the buoyant force acting on the buoy, $W$ is the weight of the buoy, and $T$ is the tension in the cable.

- For equilibrium it follows that:
  $$T = F_B - W$$
  $$F_B = \gamma V = 10.1 \times \left[(\pi/6)(1.5)^3\right] = 1.785 \times 10^4 \text{ N}$$
  $$T = 1.785 \times 10^4 - 0.85 \times 10^4 = 9.35 \text{ kN}$$

*The net effect of the pressure forces on the surface of the buoy is equal to the upward force of magnitude $F_B$.*
Example

- The figure shows a metal part (object 2) hanging by a thin cord from a floating wood block (object 1). The wood block has a specific gravity \( S_1 = 0.3 \) and dimensions of \( 50 \times 50 \times 10 \) mm. The metal part has a volume of 6,600 mm\(^3\).

- Find the mass \( m_2 \) of the metal part and the tension \( T \) in the cord.
First draw the free-body diagrams.
Example

Sum forces on the block:

\[ T = F_{B_1} - W_1 \]

The buoyant force on the floating block is \( F_{B_1} = \gamma \mathcal{V}_{D_1} \), where \( \mathcal{V}_{D_1} \) is the submerged volume:

\[ F_{B_1} = \gamma \mathcal{V}_{D_1} = (9800 \text{ N/m}^3)(50 \times 50 \times 7.5 \text{ mm}^3)(10^{-9} \text{ m}^3/\text{mm}^3) \]
\[ = 0.184 \text{ N} \]

The weight of the block is

\[ W_1 = \gamma S_1 \mathcal{V}_1 = (9800 \text{ N/m}^3)(0.3)(50 \times 50 \times 10 \text{ mm}^3)(10^{-9} \text{ m}^3/\text{mm}^3) \]
\[ = 0.0735 \text{ N} \]

Hence the tension on the cord is

\[ T = (0.184 - 0.0735) = 0.110 \text{ N} \]
Example

Apply force equilibrium to the metal part:

\[ W_2 = T + F_{B2} \]

Because the metal part is submerged, use the volume of the part to calculate the buoyant force:

\[ F_{B2} = \gamma V_2 = (9800 \, \text{N/m}^3)(6600 \, \text{mm}^3)(10^{-9}) = 0.0647 \, \text{N} \]

Hence, the weight is given by \( W_2 = (0.110 + 0.0647) = 0.175 \, \text{N} \), and the mass is found from

\[ m_2 = \frac{W_2}{g} = 17.8 \, \text{g} \]
Stability

- A body is said to be in a stable equilibrium position if, when displaced, it returns to its equilibrium position.

- Conversely, it is in an unstable equilibrium position if, when displaced (even slightly), it moves to a new equilibrium position.
Stability
Stability

- Stability considerations are particularly important for submerged or floating bodies since the centers of buoyancy and gravity do not necessarily coincide. A small rotation can result in either a restoring or overturning couple.

- **Center of Gravity** is the point in a body where the gravitational force may be taken to act.

- **Center of Buoyancy** is the center of the gravity of the volume of water which a hull displaces.
Stability

- For floating bodies the stability problem is more complicated, because as the body rotates the location of the center of buoyancy may change.

- In the figure, as the body rotates the buoyant force, $F_B$, shifts to pass through the centroid of the newly formed displaced volume and, as illustrated, combines with the weight, $W$, to form a couple, which will cause the body to return to its original equilibrium position.
Stability

- For the relatively tall, slender body shown in the figure, a small rotational displacement can cause the buoyant force and the weight to form an overturning couple.
Stability

- A floating body may still be stable when G is directly above B. This is because the centroid of the displaced volume shifts to the side to a point B’ during a rotational disturbance while the center of gravity G of the body remains unchanged. If point B’ is sufficiently far these two forces create a restoring moment and return the body to the original position.
Stability

- A measure of stability for floating bodies is the **metacentric height** $GM$, which is the distance between the center of gravity $C$ (or $G$ or $CG$) and the metacenter $M$—the intersection point of the lines of action of the buoyant force through the body before and after rotation.

- A floating body is stable if point $M$ is above point $G$ ($GM$ is positive).

- A floating body is unstable if point $M$ is below point $G$ ($GM$ is negative).
Stability

- The weight and the buoyant force acting on the tilted body generate an overturning moment instead of a restoring moment, causing the body to **capsize** (overturn)

- The length of the metacentric height GM above G is a measure of the stability: the larger it is, the more stable is the floating body
Stability

\[ CM = \frac{I_{oo}}{V} \]

\[ GM = CM - CG \]

Metacentric height = \( GM = \frac{I_{oo}}{V} - CG \)
A block of wood 30 cm square in cross section and 60 cm long weighs 318 N. Will the block float with sides vertical as shown?
**Example**

**Solution**  First determine the depth of submergence of the block. This is calculated by applying the equation of equilibrium in the vertical direction.

\[
\Sigma F_y = 0
\]

- weight + buoyant force = 0

\[-318 \text{ N} + 9810 \text{ N/m}^3 \times 0.30 \text{ m} \times 0.60 \text{ m} \times d = 0\]

\[
d = 0.18 \text{ m} = 18 \text{ cm}
\]

Determine whether the block is stable about the longitudinal axis:

\[
GM = \frac{I_{00}}{V} - CG = \frac{\frac{1}{12} \times 60 \times 30^3}{18 \times 60 \times 30} - (15 - 9)
\]

\[
= 4.167 - 6 = -1.833 \text{ cm}
\]
Example

Because the metacentric height is negative, the block is not stable about the longitudinal axis. Thus a slight disturbance will make it tip. Next, check to see if the block is stable about the transverse axis:

\[ GM = \frac{\frac{1}{12} \times 30 \times 60^3}{18 \times 30 \times 60} - 6 = 10.67 \text{ cm} \]

The block is stable about the transverse axis and will float with the short sides vertical.